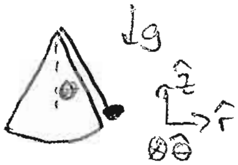
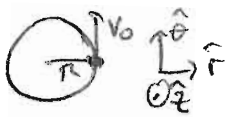
Phys 261 Hwk #8

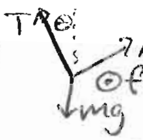
Morin 3.70





top view

A mass (m) executes circular (R) motion while suspended (g) from a rope and while scraping against a cone with friction (μ). If the initial velocity is v_0, we seek the time until the motion stops.

Forces: 

$\Longrightarrow \vec{F} = (-T\sin\theta + N\cos\theta)\hat{r} - f\hat{\theta} + (T\cos\theta + N\sin\theta - mg)\hat{z}$

Equating with $m\vec{a}$ we learn:

$$-T\sin\theta + N\cos\theta = -mv^2/R \qquad (\hat{r})$$
$$T\cos\theta + N\sin\theta = mg \qquad (\hat{z})$$
$$mR\ddot{\theta} = m\dot{v} = -f \qquad (\theta)$$

Solving $(\hat{r})$ & $(\hat{z})$ for N:

$$\cos\theta(\hat{r}) + \sin\theta(\hat{z}) \Longrightarrow N = -\frac{mv^2}{R}\cos\theta + mg\sin\theta$$

And since this is sliding friction we know $f = \mu N$, so we get this eqn of motion:

$$\dot{v} = \frac{-f}{m} = \frac{\mu\cos\theta}{R}v^2 - \mu g\sin\theta \equiv \alpha v^2 - \beta \qquad \begin{pmatrix}\alpha \equiv \mu\cos\theta/R \\ \beta \equiv \mu g\sin\theta\end{pmatrix}$$

Notice BTW that there is a special speed (of $\sqrt{gR\tan\theta}$) where $\dot{v} \to 0$. That is of course the speed above which the mass doesn't actually scrape the cone. Assuming v_0 is less than this, the time to stop is then

$$T = \int_{v_0}^{0} \frac{dv}{\dot{v}} = \int_0^{v_0} \frac{dv}{-\dot{v}} = \int_0^{v_0} \frac{dv}{\beta - \alpha v^2} = \frac{1}{\beta}\int_0^{v_0} \frac{dv}{1 - (\frac{\alpha}{\beta})v^2}$$

change to $w \equiv \sqrt{\frac{\alpha}{\beta}}\,v$
$\Rightarrow dv = \sqrt{\frac{\beta}{\alpha}}\,dw$
define $w_0 \equiv \sqrt{\frac{\alpha}{\beta}}\,v_0$

$$= \frac{1}{\sqrt{\alpha\beta}}\int_0^{w_0} \frac{dw}{1-w^2} = \frac{1}{\sqrt{\alpha\beta}}\,\text{arctanh}(w_0)$$

Appendix A.27 for integral

$$= \frac{1}{\sqrt{\alpha\beta}}\,\text{arctanh}\left[\sqrt{\frac{\alpha}{\beta}}\,v_0\right] = \frac{1}{\mu}\sqrt{\frac{R}{g\sin\theta\cos\theta}}\,\text{arctanh}\left[v_0\sqrt{\frac{\cos\theta}{gR\sin\theta}}\right]$$

note this argument is ≤ 1 given our assumption about v_0, which is good b/c the range of tanh is -1 to 1.

This is a little messy, as advertised, but passes a few checks:

- when $\mu \to 0$, $T \to \infty$ as we expect, i.e. we never slow down
- when $\theta \to \frac{\pi}{2}$ (and $\sin\theta \to 1$ while $\cos\theta \ll 1$) we expand $\text{arctanh}(\epsilon) \simeq \epsilon + \cdots$ to find

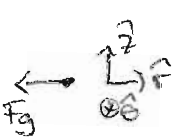
$$T \to \frac{1}{\mu}\sqrt{\frac{R}{g\cos\theta}}\cdot\left(v_0\sqrt{\frac{\cos\theta}{gR}}\right) \to \frac{v_0}{\mu g}$$

, exactly as expected for a flat table where the friction force is just μmg, independent of speed.

Morin 5.56 A planet made of rubble has density ρ. How fast can it spin before material flies off at the equator? I.e. For what ω is the gravity force just large enough to keep a pebble in orbit at the surface R?

$\vec{F} = m\vec{a}$ tells us

$$-\frac{GMm}{R^2}\hat{r} = -m\omega^2 R\hat{r} \Longrightarrow \omega = \sqrt{\frac{GM}{R^3}}$$

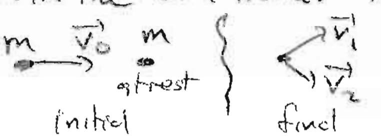


Phrasing the result in terms of period and density $\rho = \frac{M}{\frac{4}{3}\pi R^3}$,

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{G\rho}}$$

which the same period as in "Speedy Travel" through tunnels, and is ~ 1.4 hours for Earth density.

Morin 5.81 We are asked to show that in a standard 2D elastic collision between equal mass pool balls, the outgoing velocities are perpendicular, this time analyzing the collision in the CM frame. First let's recall the argument purely in the lab frame:

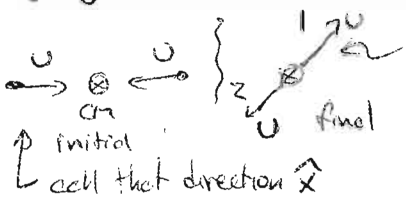


conservation laws: $m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2$

$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$

The momentum law says $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$, i.e. these vectors form a triangle. The energy law says it's the sort of triangle for which the Pythagorean theorem holds, i.e. a right triangle.

Now in the CM frame the "story" is this:



where $u = \frac{v_0}{2}$ is also the speed of the CM relative to the lab. The rule for translating the CM velocities to the lab frame is to add the CM velocity, $\vec{V}_{cm} = u\hat{x}$, so...

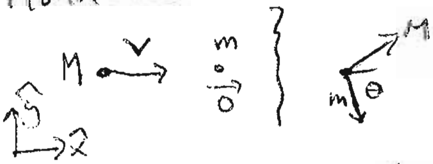
$$\vec{v}_1^{\,lab} = \vec{v}_1^{\,cm} + \vec{V}_{cm} = u\hat{e} + u\hat{x} = u(\hat{e}+\hat{x})$$

$$\vec{v}_2^{\,lab} = \vec{v}_2^{\,cm} + \vec{V}_{cm} = -u\hat{e} + u\hat{x} = u(-\hat{e}+\hat{x})$$

But then the dot product is:

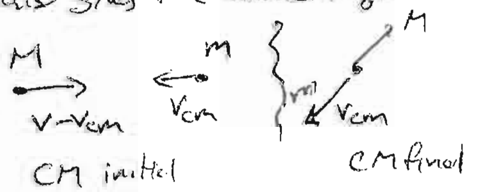
$$\vec{v}_1^{\,lab} \cdot \vec{v}_2^{\,lab} = u^2(\hat{e}+\hat{x})\cdot(-\hat{e}+\hat{x}) = u^2\left[-1 + \hat{e}\cdot\hat{x} - \hat{x}\cdot\hat{e} + 1\right] = 0 \Rightarrow \vec{v}_1 \perp \vec{v}_2$$

Morin 5.83



Another collision with a stationary target, this time with unequal masses, and we are asked to find the direction θ of the mass m in the case where its $|v_y|$ is as large as possible.

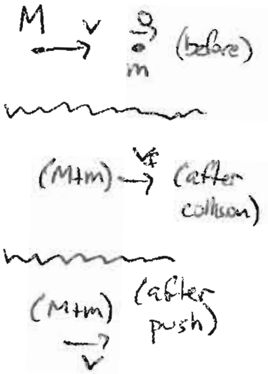
The key insight is that since the CM moves purely in the $\hat{x}$ direction, the "translation" of velocity vectors between CM & lab frames does not affect v_y. Thus the scattering process which gives the largest $|v_y|$ in the ~~lab~~ CM frame also gives the same largest $|v_y|$ in the lab frame. But in the CM frame the process is easy to describe: M & m approach with opposite momenta, then scatter in some new direction with exactly the same speeds, assuming the collision is elastic. In particular the mass m comes in at v_{cm}, and goes out at v_{cm}.



To maximize $|v_y|$ we plainly want m to scatter straight down the $\hat{y}$ axis (in the CM frame). Then translating back to the lab frame by adding $\vec{V}_{cm}$ to this, we find the direction:

(See the Math notebook for a solution purely in terms of lab frame quantities.)

Morin 5.89 One more collision, this one inelastic. A plate (mass M) moving at v picks up a stationary pea, which slides along the plate for a while before coming to rest (relative to the plate). Now we are asked: how much work must one supply if one wants to push the plate+pea back up to speed v?



Momentum conservation gives us the speed of the system after the collision:

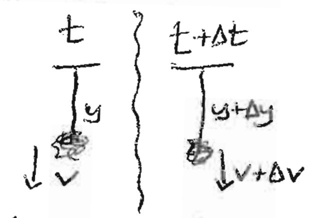
$$Mv + 0 \stackrel{!}{=} (M+m)v_f \implies v_f = v_{cm} = \frac{M}{M+m}v$$

The energy we need to supply is then this differece: (multiplying by $1 = \frac{M+m}{M+m}$ in preparation for combining)

$$W \equiv \frac{1}{2}(M+m)v^2 - \frac{1}{2}(M+m)v_f^2 = \frac{1}{2}\frac{(M+m)^2}{M+m}v^2 - \frac{1}{2}\frac{M^2}{M+m}v^2 = \frac{1}{2}mv^2 \cdot \frac{2M+m}{M+m}$$

Note that the factor multiplying the $\frac{1}{2}mv^2$ is always bigger than $\underline{1}$, which makes sense in that between the first frame and the last we've added one particle with energy $\frac{1}{2}mv^2$. The work we do is greater, though, because some energy is lost to internal degrees of freedom in the collision. Note that when $M \to \infty$, the factor is 2 and the amount lost is exactly $\frac{1}{2}mv^2$. The "intuitive" explanation is that in that limit the plate frame <u>is</u> the CM frame, and the process is described as: pea was moving at $(-v)$, then scraped to a stop, losing all $\frac{1}{2}mv^2$ of its kinetic energy to heat.

Morin 5.91 You hold on to one end of a chain (length L, mass σL) and drop it as one big ball. As the ball falls and the chain straightens out, we seek $F(t)$, the force your hand has to supply to keep the end in place.



Note that as the chain falls we know $v = gt$ and therefore $y = \frac{1}{2}gt^2$

In the classic Δt method we make a "movie", and choose as our system that little piece of chain which stops between t and $t+\Delta t$. Its length is $v\Delta t \equiv \Delta y$ where we are told to assume the ball falls freely, i.e. $v = gt$. The downward momentum is then $\Delta m\, v$ where $\Delta m = \sigma \Delta y$.

The upward force which your hand must supply to stop the links is then

$$\frac{\Delta p}{\Delta t} = \sigma \frac{\Delta y}{\Delta t} v = \sigma v^2 = \sigma g^2 t^2$$

To this we add the weight $(\sigma y)g$ of the chain already at rest to get the total:

$$F(t) = \sigma g^2 t^2 + \sigma\left(\tfrac{1}{2}gt^2\right)g = \tfrac{3}{2}\sigma (gt)^2$$

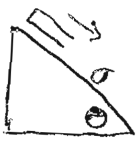
Note that we run out of chain at a time when $\frac{1}{2}gt^2 = L$, so F peaks at $3(\sigma L)g$ just before dropping to a steady $(\sigma L)g$ for a stationary chain.

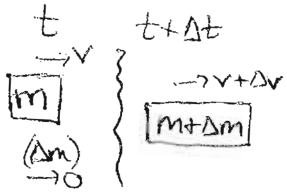
Also note that since there is only one moving part to this problem, one could also take the entire chain as the system and then write the momentum as

$$p(t) = \sigma\underbrace{\left[L - \tfrac{1}{2}gt^2\right]}_{\text{the length in motion}}\underbrace{(gt)}_{\text{at this speed}} = \sigma L g t - \tfrac{1}{2}\sigma g^2 t^3$$

So the total downward force on the system is $F_{ext} = \frac{dp}{dt} = \sigma L g - \frac{3}{2}\sigma g^2 t^2$ which we recognize as gravity plus the upward force of your hand.

Morin 5.94 An "essentially massless" dustpan starts from the top of a ramp, sliding or rolling down w/o friction except for encountering a layer of dust which it picks up as it moves. σ gives the amount of dust per length ($[\sigma] = M/L$). We seek the motion, starting at rest with an empty dustpan.





Making a Δt movie, the story is that every Δt the dustpan picks up some $\Delta m = \sigma \Delta x = \sigma v \Delta t$ while some external forces act. Calling x the distance along the incline, the mass $m(t)$ is the amount of dust in the pan, i.e. $m(t) = \sigma x(t)$

$$\Longrightarrow \Delta p = (m+\Delta m)(v+\Delta v) - mv = m\Delta v + (\Delta m)v + (\text{irrelevant } \Delta^2 \text{ term})$$
$$= \left[(\sigma x)\dot{v} + \sigma v^2\right]\Delta t$$

Equating $\Delta p/\Delta t$ with the net external force $F_{ext} = mg\sin\theta$ in our $\hat{x}$ direction we then get this nasty looking differential equation:

$$\sigma x \ddot{x} + \sigma \dot{x}^2 = (\sigma x) g \sin\theta$$

See the Mathematica notebook for a fruitless attempt to get the solution. Then thinking about chapter 1 considerations, realize that the solution will be some expression for $x(t)$ involving all the building blocks we have: σ, g, θ and t. But dimensional analysis says the only way to build a length quantity is gt^2. So take as our ansatz the form

$$x_A = \tfrac{1}{2}at^2 \implies \dot{x}_A = at \implies \ddot{x}_A = a$$

which when plugged in to the diff eq gives:

$$\left(\tfrac{1}{2}at^2\right)(a) + (at)^2 = \left(\tfrac{1}{2}at^2\right) g\sin\theta$$

which will work just fine if $a = \frac{1}{3} g\sin\theta$

Note that Morin drops a hint by asking for "the acceleration" and not $a(t)$ or $x(t)$. Also, if it seems strange that the precise amount of σ doesn't matter (after all, what happens when $\sigma \to 0$?!) note that the "essentially massless" condition means that we can neglect the dustpan weight m_0 in writing $m = m_0 + \sigma x$. That is, the more complete answer would be the solution to the harder eqn $(\sigma x + m_0)\ddot{x} + \sigma(\dot{x})^2 = (\sigma x + m_0) g\sin\theta$ which will involve the length m_0/σ, and we are observing here that if one waits long enough that x is large compared to this, then $x(t) \approx \frac{1}{2}at^2$