Comments on Phys 261 Midterm #2

So I gave a solution, but what were some of the ways one could stumble along the way?

1(a) When drawing the $\hat{r}\,\hat{\theta}$ basis vectors, note that $\hat{r}$ always points at the object we're tracking (some folks drew $\hat{\theta}\!\uparrow\!\!\rightarrow\hat{r}$, as if they were $\hat{x}\,\hat{y}$), and that $\hat{\theta}$ points in the direction the object moves when θ increases. Equationwise, we always claim that $\frac{d}{dt}\hat{r} = \dot{\theta}\hat{\theta}$, i.e. if $\dot{\theta}$ is positive, then $\hat{r}$ is swinging in the $+\hat{\theta}$ direction. Make sure your geometry reflects this, or you'll flip a sign somewhere. Concerning resolving vectors, it happens again and again that we write $F_{x \text{ or } y} = (\text{something})(\pm \text{sin or cos})$. If unsure, check by asking what happens when $\theta \to 0$ or $\pi/2$, noting that $\cos(\theta)$ is the even one with $\cos(0)=1$, while $\sin\theta$ is the odd one with $\sin(0)=0$.

(b) When asked to find the differential equation satisfied by $\theta(t)$, many people responded with expressions of the form $\ddot{\theta} = (\text{expression involving } N \text{ or } f)$. While those equations may be true, they are not anything we can attempt to solve until the unknowns like N & f are eliminated in favor of some expression involving θ & $\dot{\theta}$ (and known parameters). That is, in chapter 3 we learned how to take a given system and solve for all the unknown forces and obtain expressions for various accelerations. We always begin with that, but now we are taking the next step (when possible!) and attempting to solve the resulting differential equations. In this problem of course, the diff eq is too hard to solve.

(c) Some people thought that when $\mu=0$ the equation could be solved, but no, $\ddot{\theta} = -\frac{g}{R}\sin\theta$ is the hard one which we frequently beg off solving exactly by saying suppose $\theta(t)$ is alway so small that $\sin\theta \approx \theta$. $\ddot{\theta} = +\frac{g}{R}\sin[\theta(t)]$ is equally hard, but describes our falling bead, where θ will unfortunately not stay small.

If one blithely wrote integral signs, saying e.g. $\int\ddot{\theta} = \int\sin\theta \Rightarrow \dot{\theta} = -\cos\theta$, then one was confusing the integral dt with an integral $d\theta$. The true statement is $\ddot{\theta}(t) = \sin(\theta(t)) \Rightarrow \ddot{\theta}\,\dot{\theta} = \sin\theta\,\dot{\theta}$ and we are on our way to rederiving the work-energy $\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2(t)\right] = \frac{d}{dt}\left[-\cos\theta(t)\right]$ theorem, which is of course what we use to figure out how fast the bead is moving at a given θ.

2. Most people had no trouble finding that if $\dot{v} = -\alpha v^3$, then α has units $[\alpha] = \left[\frac{\dot{v}}{v^3}\right] = \frac{T}{L^2}$. Many people then thought that one might be able to solve a differential equation by unit analysis. Alas, not so. Units tell one that if $v = Ae^{-\gamma t}$ then A had better be a velocity (in fact $v(0)=A$) and $[\gamma] = \frac{1}{T}$ so the argument of the exponential is dimensionless. Likewise if $v = B(1+\beta t)^n$ then B is again the starting velocity and $[\beta] = \frac{1}{T}$. These are good, helpful observations we can use at the end to make sure we haven't committed some algebra error along the way when we write $\beta = $ expression involving α, v_0. Some people made algebra errors, noticed as they proceeded that the units were no longer matching and erroneously concluded that the proposed ansatz must not work. Starting from dimensionally correct equations and doing legal algebra steps will always lead to more dimensionally reasonable relations, so the correct conclusion is that there must be an algebra mistake. Here are a few missteps taken in each case by more than one person:

$$(1+\beta t)^n \overset{?!?}{=} 1^n + (\beta t)^n \qquad \left[(1+\beta t)^n\right]^3 \overset{?!?}{=} (1+\beta t)^{n+3} \qquad \frac{(1+\beta t)^{3n}}{(1+\beta t)^n} \overset{?!?}{=} (1+\beta t)^3$$

Anyone can make a silly misstep; the trick is to catch it (e.g. with "checks") and go back and find it.

3. (a) Many people drew pictures which showed that they understood that $\vec{V}_{cm}$ is a vector pointing at 45° in the $(\hat{x}+\hat{y})$ direction, but they ignored their drawings, the presence of $\hat{x}$ & $\hat{y}$ on the problem statement, and the request for the vector $\vec{V}_{cm}$ to write thing like $V_{cm} = \frac{P}{M} = \frac{2mv_0}{2m}$. The correct thing to do is use the language of vectors as we write $\vec{V}_{cm} = \frac{\vec{P}}{M} = \frac{1}{2m}\left[mv_0\hat{x} + mv_0\hat{y}\right] = \frac{v_0}{2}(\hat{x}+\hat{y})$

BTW the speed (magnitude of $\vec{v}$) of the CM is $V_{cm} = |\vec{V}_{cm}| = \sqrt{(\frac{v_0}{2})^2+(\frac{v_0}{2})^2} = \frac{v_0}{\sqrt{2}}$

And likewise to get the speed of one of the particles in the CM frame one has to form the velocity vector first: $\vec{v}_1^{\,cm} = \vec{v}_1^{\,lab} - \vec{V}_{cm}^{\,lab} = v_0\hat{x} - \frac{v_0}{2}(\hat{x}+\hat{y}) = \frac{v_0}{2}(\hat{x}-\hat{y}) \Rightarrow |\vec{v}_1^{\,cm}| = \frac{v_0}{\sqrt{2}}$

(b) Some people correctly and reasonably declared that the energy lost in the collision was $\Delta K = 2\cdot\frac{1}{2}m(v_0^2 - v_f^2)$ but then left it there without figuring out that $v_f = V_{cm} = \frac{v_0}{\sqrt{2}}$.

Again, chapter 1 is the most important: what's known, & what's not known, & what will the answer look like

(c) The most important thing to understand about this one is that there are three unknowns (the speed v which we seek, and $\vec{v}_2$ for the unobserved particle) and three equations to be solved, so there is a definite answer (or two, as happens with quadratics). Some (many!) people got entered in thickets of arcsin's. The best advice when solving systems of equations is: solve simple equations first, so don't write $\phi = \arcsin(\text{stuff})$ and then plug in to an expression involving $\cos\phi$. Instead get rid of the trigonometry and ϕ all at once by solving for $\sin\phi = (\cdots)$ and $\cos\phi = (\cdots)$ and then forming $\sin^2\phi + \cos^2\phi = 1 = (\cdots)^2 + (\cdots)^2$

↑ expression of v_0, v & v_2

Some people claimed the angle ϕ (of $\vec{v}_2$) was "obvious", and so didn't actually solve. But to some it was obvious that the direction was perpendicular to $\vec{v}_1$, or that $\vec{v}_1$ and $\vec{v}_2$ are symmetric about $\pi/4$, or that $\vec{v}_2$ is in the $\hat{x}$ or the $\hat{y}$ direction. There is no merit to any of these assumptions, but as it happens $\vec{v}_2$ is perpendicular to $\vec{v}_1$. It is also true that in the collision of one moving pool ball with a stationary object ball, the outgoing velocities will be perpendicular. Our problem is different from that one, and there is no universal principle that pool balls always emerge from a collision at right angles. Nevertheless, the special set up of our problem also happens to lead to right angles, as can be quickly seen from the results of part (a), when you realize that $|\vec{V}_{cm}^{\,lab}| = |\vec{v}_1^{\,cm}| = |\vec{v}_2^{\,cm}|$, so $\vec{v}_1^{\,lab} = \vec{v}_1^{\,cm} + \vec{V}_{cm}^{\,lab}$ and $\vec{v}_2^{\,lab} = \vec{v}_2^{\,cm} + \vec{V}_{cm}^{\,lab}$ are the sums and differences of two unit vectors; (the same speed) and those are always perpendicular: $\vec{v}_1 = v(\hat{e}_1 + \hat{V}_{cm})$ $\vec{v}_2 = v(-\hat{e}_1 + \hat{V}_{cm}) \Rightarrow \vec{v}_1\cdot\vec{v}_2 = 0$

But as I said, it's best not to worry about particle 2 at all and just eliminate ϕ first before v_2.

BTW since the initial p_x and p_y are both mv_0, many people took their first step of algebra to write $p_x^{final} = p_y^{final}$. This amounts to "solving" for the known v_0 in one equation and then plugging in to the other equation. But v_0 is not some we want to eliminate, it is a known input. Better to use equations to eliminate unknowns.

4. The core process in a "rocket" problem is to identify a "system" and then observe that any momentum change comes from a net force on that system. To get there we make a movie showing all the moving pieces in our system at generic time t and a moment later $(t+\Delta t)$ after some of the pieces may have interacted. It is crucial that the "system" have exactly the same pieces before and after. If one starts with total mass $M(t)$ at time t, you will go wrong drawing an M and a Δm in the later frame. Likewise, some people had a rocket of mass M plus fuel (Δm) in the first frame, and a rocket of mass $(M-\Delta m)$ plus a Δm in the 2nd frame, i.e. losing a mass Δm between frames. The most common error is to confuse the generic t of the first "frame" with $t=0$, and claim that $v=0$ or $M=m_0$. Correctly, we just label the current mass & speed of the rocket as $m(t)$ and $v(t)$ without claiming to know them right away. Here we do know $m(t)=m_0-bt$ as we burn away fuel, but we never know $v(t)$. After all if $v(t)=v_0$, then we are done — we claim to know that at a generic time t, the velocity is alway the constant v_0. In terms of expressions like $v(t)$ and $v(t+\Delta t)$ (or $v+\Delta v$) we express the change in momentum $\Delta p \equiv p(t+\Delta t)-p(t) =$ (expression involving Δm's and Δv's among other things). If your expression for Δp has finite term (i.e. with no Δ's), go check the labelling again — the next step is to divide by Δt and let $\Delta t \to 0$, and a term like $m/\Delta t$ will blow up to infinity. Unfortunately, some will invent a rule that e.g. $\frac{M}{\Delta t} \to \dot{m} = b$ and wander off the path. Having found $\frac{dp}{dt}$ $(= M(t)\dot{v} - bu$ here$)$ we equate with the net external force, which here is the force of gravity, $-M(t)g$. A common error is to be unclear about which mass to use. Some people claimed the force is always $M_0 g$, some people claimed the inertia was M_0 even when the gravitating mass was $M(t)$. Others used the symbols M, m and M_0 interchangeably, without ever noting that $M(t)=m_0-bt$. It is totally fine to introduce shorthands like M along the way, but the key to staying on track is to be sure you know how to express the shorthand in terms of known inputs. In the current problem the last step asks for an integral like $\int \frac{dt}{M}$ and then it matters what function $M(t)$ is. If one mislabelled one of the velocities e.g. saying that the ejected fuel starts at rest (instead of $v(t)$) or that its later velocity is u (instead of $v-u$ upward), one ended up with a diffeqn involving v, e.g. $M\dot{v} = -bv + bu - Mg$. If you ignored the remark about $\dot{v}$ being a function of time explicitly, i.e. not implicitly through an unknown v, and continued on to integrate, then you faced the need to integrate $\int dt \frac{v(t)}{M(t)}$. In truth one can't do anything with this since $v(t)$ is unknown at this point. Some people made "progress" by ignoring the time dependence in v and M and writing $\int \frac{v}{M} dt = \frac{v}{M} \cdot t$, thus treating $v(t)$ as a constant en route to finding that $v(t)$ is some linear function of t. Others ignored only the time dependence of M and wrote $v = \int \dot{v}\, dt =$ (constants)$\cdot \int v(t)\,dt =$ (constants) $x(t)$. This is not really progress, since $x(t)$ is not something we know. One is just trading the diffeqn $\dot{v} =$ (something involving v) for the diffeqn $v = \dot{x} =$ (something involving x).