Lecture 3: We covered Shankar 1.6 ($\geq$ page 26) and 1.9

We check that $(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$. Let

$$\Lambda|V\rangle = |V'\rangle \text{ and } \Omega|V'\rangle = |V''\rangle .$$

By the definition of adjoint operators we have

$$\langle V'| = \langle V|\Lambda^\dagger \text{ and } \langle V''| = \langle V'|\Omega^\dagger .$$

Thus we obtain

$$\langle V''| = \langle V'|\Omega^\dagger = \langle V|\,\Lambda^\dagger\Omega^\dagger .$$

We compare this with

$$|V''\rangle = \Omega|V'\rangle = (\Omega\Lambda)|V\rangle$$

and its adjoint

$$\langle V''| = \langle V|(\Omega\Lambda)^\dagger$$

where we treat the product as a single operator to obtain the required result.

Unitary operators If $\{|i\rangle\}$ and $\{|j'\rangle\}$ are two independent basis sets the operator that takes us from the unprimed to the primed basis[1]

$$U \equiv \sum_{\ell=1}^{n} |\ell'\rangle\,\langle\ell| \text{ with matrix elements } \quad U_{ij} = \langle i|j'\rangle$$

in the unprimed basis.

$$\text{Therefore,} \quad U^\dagger_{j\ell} = U^*_{\ell j} = \langle j'|\ell\rangle$$

and clearly $\sum_j U_{ij}\,U^\dagger_{j\ell} = \delta_{i\ell}$.

Be clear that the columns of a unitary matrix form an orthonormal basis: the jth and j'th column given by U_{ji} and $U_{\ell i}$ are orthogonal because $\sum_{i'}\langle j|i'\rangle\langle i'|\ell\rangle = \delta_{j\ell}$. The i^{th} row of U is U_{ij} with j varying from 1 to n. The rows are orthogonal, i.e, $\sum_j U_{ij}\,U^*_{i'j}$ should vanish for $i \neq i'$. This is evident since we are computing $(UU^\dagger)_{ii'}$.
Please look at the unitary matrix that diagonalizes the Pauli matrix σ_y and verify the above properties.

Lecture 4 (Pages 35-44)

[1] We can write the following in the form

$$U = \sum_i\sum_j |i\rangle\,U_{ij}\,\langle j| = \sum_i\sum_j |i\rangle\langle i|j'\rangle\langle j| = \sum_{j=1}^{n} |j'\rangle\langle j|$$

Lecture 5

Two Hermitian operators Ω and Λ can be simultaneously diagonalized (i.e., there exists a basis in which both operators only have diagonal matrix elements) if and only if they commute.

This is all you need to remember and be able to use. If you wish you may go through the proof in Shankar. The following 2×2 calculation illustrates the key ideas. First assume that the eigenvalues of Λ are non-degenerate. Since it is a Hermitian matrix there exits a basis in which it is diagonal:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} .$$

In this basis let Ω be given by

$$\Omega = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} .$$

Compute $\Omega\Lambda$ and $\Lambda\Omega$; since the operators commute these are equal:

$$\Omega\Lambda = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} a\lambda_1 & c\lambda_2 \\ c^*\lambda_1 & b\lambda_2 \end{pmatrix} .$$

$$\Lambda\Omega = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \begin{pmatrix} a\lambda_1 & c\lambda_1 \\ c^*\lambda_2 & b\lambda_2 \end{pmatrix} .$$

Comparing the 12 matrix element we have $c = 0$ is $\lambda_1 \neq \lambda_2$. Therefore, we have shown that if Λ is non-degenerate we can diagonalize Λ and ω simultaneously if they commute.

We cannot conclude this if Λ is degenerate. Degeneracy makes proofs harder. Now we use the following artifice. Note that in the diagonal representation the matrix Λ with two degenerate eigenvalues has the form,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \lambda_1 I.$$

Note that any change of basis (one orthonormal basis to another) leaves the identity matrix invariant. Be absolutely clear about this. Since Ω is Hermitian we can make a change of basis to make it diagonal. Now do the same change basis on Λ which remains diagonal. Ergo, both matrices are diagonal in this basis.

We did example 1.8.6 on page 46 from Shankar in detail. Please read this example in detail. Here are a few comments. What is $|x(t)\rangle$? It represents an abstract vector, a ket, for a given value of the time t and as t evolves we get a (continuous) sequence of vectors. Its second derivative $(d^2/dt^2)|x(t)\rangle$, denoted by $|\ddot{x}(t)\rangle$ in Shankar is obtained as follows: Since $|x(t)\rangle$ corresponds to the column vector

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

for the given normal mode problem we have

$$|\ddot{x}(t)\rangle \leftrightarrow \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} .$$

The dynamical equation is

$$|\ddot{x}(t)\rangle = \Omega\,|x(t)\rangle .$$

First find the eigenvalues and eigenvectors of Ω. Since Ω is symmetric (Hermitian) the eigenvectors form an orthonormal set.

$$\Omega|I\rangle \;=\; -\omega_I^2|I\rangle \;\;\text{and}\;\; \Omega|II\rangle \;=\; -\omega_{II}^2|II\rangle\,.$$

Expand $|x(t)\rangle$ in terms of the eigenvectors:

$$|x(t)\rangle \;=\; x_I(t)\,|I\rangle \;+\; x_{II}(t)\,|II\rangle\,.$$

Substitute into the dynamical equation. The left- and right-hand sides yield

$$\ddot{x}_I(t)\,|I\rangle \;+\; \ddot{x}_{II}(t)\,|II\rangle\,.$$

$$-\omega_I^2 x_I(t)\,|I\rangle \;-\; \omega_{II}^2 x_{II}(t)\,|II\rangle$$

using the eigenvalue equations. Thus we can identify

$$\ddot{x}_A(t) \;=\; -\omega_A^2 x_A(t)$$

where $A = I$ or II. The equations are decoupled and we obtain

$$x_I(t) \;=\; x_I(0)\,\cos\omega_I t \;\;\text{and}\;\; x_{II}(t) \;=\; x_{II}(0)\,\cos\omega_{II} t$$

where we have used the initial conditions, setting the initial velocities to be zero. Thus we find the final answer

$$|x(t)\rangle \;=\; x_I(0)\,\cos\omega_I t|I\rangle \;+\; x_{II}(0)\,\cos\omega_{II} t|II\rangle\,.$$

We can find (make sure you are clear about this) the coefficients using

$$x_I(0) \;=\; \langle I|x(0)\rangle \;\;\text{and}\;\; x_{II}(0) \;=\; \langle II|x(0)\rangle\,.$$

Please read the recipe given in the book on page 49 which summarizes the results of the above derivation.

Infinite-dimensional spaces

Read the book's clear description of going from a discrete set of points to describe the displacement of a string to the continuum description. So one views $f(x)$ as the components of the function (represented by the abstract ket $|f\rangle$.) Explain clearly what it means to associate a function $f(x)$ with a ket.

We start from

$$|f\rangle \;=\; \int_a^b dx\, |x\rangle\langle x|f\rangle \;=\; \int_a^b dx\, |x\rangle\, f(x).$$

The basis set $\{|x\rangle\}$ satisfies two key properties:

$$\int_a^b dx|x\rangle\langle x| \;=\; I\,.$$

$$\langle x|x'\rangle \;=\; \delta(x-x')\,.$$

We discussed some properties of the Dirac delta function.