Lecture 1: We covered Shankar 1.1 and 1.2

Some additional remarks: We included the axiom $1 \cdot |V\rangle = |V\rangle$. This is needed in the problem below.

Show that $|0\rangle = 0|V\rangle$ *for any vector* $|V\rangle$.}
Solution: There are many ways of proving such relations. One can assume the uniqueness of the null vector and do the following:

$$|V\rangle + 0|V\rangle = 1|V\rangle + 0|V\rangle = (1+0)|V\rangle = 1|V\rangle = |V\rangle$$

and by the uniqueness of the null vector we have $0|V\rangle = |0\rangle$. In the first and last equalities we have used the additional axiom.
We can also start from Shankar's hint in Exerxise 1,1 and complete the proof:

$$\begin{aligned} |0\rangle &= |V\rangle + |-V\rangle = (0+1)|V\rangle + |-V\rangle = 0|V\rangle + 1|V\rangle + |-V\rangle \\ &= 0|V\rangle + (|V\rangle + |-V\rangle) = 0|V\rangle + |0\rangle =, 0|V\rangle\,. \end{aligned}$$

It is a useful mathematical exercise to identify the axioms used in each step carefully.

A. Show that the inverse is unique:
If $|V\rangle + |-V\rangle = |0\rangle$ and $|V\rangle + |-V'\rangle = |0\rangle$ we wish to show that $|-V'\rangle = |-V\rangle$. We have

$$|-V'\rangle = |-V'\rangle + |0\rangle = |-V'\rangle + |V\rangle + |-V\rangle = |0\rangle + |-V\rangle = |-V\rangle$$

where we have used the associative law and the properties of $|-V\rangle$ and $|-V'\rangle$.

Shankar Exercise 1.1.5 Show that the row vectors $(1,1,0)$, $(1,0,1)$ and $(3,2,1)$ are linearly dependent.
Solution: denoting the vectors by $|1\rangle$, $|2\rangle$ and $|3\rangle$ respectively we see immediately that

$$2|1\rangle + |2\rangle + (-1)|3\rangle = 0\,.$$

Show that $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ are linearly independent.
Solution: The standard approach is to take a linear combination of the three vectors and set them equal to zero:

$$a(1,1,0) + b(1,0,1) + c(0,1,1) = (0,0,0)\,.$$

Now check if there is a non-trivial solution to the three simultaneous equations. We obtain $a+b = 0$, $a+c = 0$, and $b+c = 0$. Subtracting the second equation from the first we have $a-c=0$ and using the third condition we obtain $a=c=0$. It follows that $a=b=c=0$ and the three vectors are linearly independent.

Lecture 2: We covered Shankar 1.3 - 1.5

Some remark on inverses

We assumed as the book does $\Omega^{-1}\Omega \, = \, \Omega\Omega^{-1} \, = I$ as the definition of the inverse of Ω assuming it exists. We will not worry about the minimal (irreducible) set of assumptions or axioms. If you care, here are a couple of remarks: suppose there is a right inverse and a left inverse and they are different. i.e.,

$$\Omega_L^{-1}\Omega \, = \, \Omega\Omega_R^{-1} \, = I\, .$$

It is easy to justify the following steps: Start with $\Omega_L^{-1}\Omega \, = \, I$. Post-multiply (i.e., multiply from the right) both sides of the equation by Ω_R^{-1}. Thus we have

$$\Omega_L^{-1}\Omega\Omega_R^{-1} \, = \, \Omega_R^{-1}\, .$$

Now on the left-hand side combine the last two operators; by definition they yield the identity and hence, $\Omega_L^{-1} \, = \, \Omega_R^{-1}$.

A different route is to follow the theorem on page 658 of the text. The first part of the theorem proves that if $\{|V_j\rangle\}$ is a basis then $\{\Omega|V_j\rangle\}$ is a basis and shows that every $|V'\rangle$ in the n-dimensional vector space arises from a unique $|V\rangle$ under the action of Ω and there is an operator Λ that acts on $|V'\rangle$ and takes it back to its unique source $|V\rangle$. Thus it is shown that

$$\Lambda|V'\rangle \, = \, |V\rangle \quad \text{and} \quad \Omega|V\rangle \, = \, |V'\rangle\, .$$

From here it is easy to to see that $\Lambda\Omega \, = \, \Omega\Lambda \, = \, I$. Start from $\Lambda|V'\rangle \, = \, |V\rangle$ and substitute $|V'\rangle \, = \, \Omega|V\rangle$. We see that $\Lambda\Omega|V\rangle \, = \, |V\rangle$ for every $|V\rangle$ in the space and thus $\lambda\Omega \, = \, I$. Reversing the order (start from $\Omega|V\rangle \, = \, |V'\rangle$) proves the other result.

All this is is superfluous for a physics course although being clear about these considerations can be helpful in learning more mathematical aspects of physics.

It is worth remembering that a finite-dimensional (square) matrix is invertible if and only if its determinant is non-vanishing. Since the determinant is the product of the eigenvalues (you should remember this also) this implies that for invertibility a matrix must have no zero eigenvalues.

Intuitively, invertibility of Ω is assured if no two distinct vectors(kets) are mapped by Ω into the same vector (ket.) If $\Omega|V_1\rangle \, = \, |W\rangle$ and $\Omega|V_2\rangle \, = \, |W\rangle$ then given $|W\rangle$ there is no way of finding an inverse uniquely. If we subtract the second equation from the first we have

$$\Omega\, (|V_1\rangle \, - \, |V_2\rangle) \, = \, |0\rangle\, .$$

If $|V_1\rangle$ and $|V_2\rangle$ are different then there is no inverse. This is the content of Theorem A.1.1 which states that Ω^{-1} exists if $\Omega|V\rangle \, = \, |0\rangle$ implies $|V\rangle = |0\rangle$.

One of the most useful results is the completeness relation (also known as the resolution of the identity). Given an orthonormal basis $\{|i\rangle\}$ we have

$$\sum_{i=1}^{n} |i\rangle\langle i| \;=\; I\,.$$

One way to think of this is in row and column notation. For the standard basis we have

$$|1\rangle \;\leftrightarrow\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \langle 1| \;\leftrightarrow\; (1, 0, 0, 0 \cdots 0)\,.$$

Thus

$$|1\rangle\langle 1| \;\leftrightarrow = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (1\,0\,0 \cdots 0) \;=\; \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

i.e., an $n \times n$ matrix whose 11 element is unity and the rest of the elements are zero. Similarly $|j\rangle\langle j|$ is a matrix with the jj element being unity and all the other elements zero. The identity follows. For another proof presented in the class see p22-23 in Shankar.

We illustrated the use of this identity.

(a) The **inner product of two vectors** can be obtained by inserting the identity operator.

$$\langle W|V\rangle \;=\; \sum_{i=1}^{n} \langle W|i\rangle\langle i|V\rangle \;=\; \sum_{i=1}^{n} w_i^* v_i\,.$$

(b) The **matrix elements of an operator** can de defined: Let $|W\rangle \;=\; \Omega|V\rangle$. We look at the i^{th} component by computing the inner product by "dotting" with $\langle i|$:

$$w_i \;=\; \langle i|\Omega|V\rangle\,.$$

Introducing the resolution of the identity we have

$$w_i \;=\; \sum_{j=1}^{n} \langle i|\Omega|j\rangle\, \langle j|V\rangle \;=\; \sum_{j=1}^{n} \Omega_{ij} v_j$$

where we have defined the matrix element, $\Omega_{ij} \;\equiv\; \langle i|\Omega|j\rangle$. This is done in Sec. 1.6 differently.

(c) **Exercise 1.7.1a**: The trace of an operator is defined as the sum of the diagonal elements. We verify for finite-dimensional operator that $Tr\Lambda\Omega \;=\; Tr\Omega\Lambda$.

$$Tr\Lambda\Omega \;=\; \sum_i \langle i|\Lambda\Omega|i\rangle \;=\; \sum_i \sum_j \langle i|\Lambda|j\rangle\, \langle j|\Omega|i\rangle$$

where we have used the completeness relation in the last step. Since this is a sum of a product of two complex numbers the order is immaterial and so we have

$$Tr\Lambda\Omega \;=\; \sum_i \sum_j \langle j|\Omega|i\rangle\, \langle i|\Lambda|j\rangle\,.$$

Now the sum on i can be replaced by the identity and so

$$Tr\Lambda\Omega \;=\; \sum_j \langle j|\Omega\Lambda|j\rangle \;=\; Tr\Omega\Lambda.$$

The completeness relation finds a multitude of uses and you should become familiar with, even adept at, using it.

Finally, here is a verification of the matrix elements of $\Omega^\dagger$ done on page 26. In a particular basis $\{|i\rangle\}$ in an n-dimensional vector space let $\Omega|i\rangle \;=\; |i'\rangle$ and by definition $\langle i'| \;=\; \langle i|\Omega^\dagger$. We have

$$\left(\Omega^\dagger\right)_{ij} \;=\; \langle i|\Omega^\dagger|j\rangle \;=\; \langle i'|j\rangle$$

where we have used the definition of $\langle i'|$. Now

$$\left(\Omega^\dagger\right)_{ij} \;=\; \langle j|i'\rangle^* \;=\; \langle j|\Omega|i\rangle^*$$

which shows that the ij^{th} matrix element of $\Omega^\dagger$ is the complex conjugate of the ji^{th} matrix element of Ω.