

7.1 The student Misty Fide comes to you and says, "I am confused by tunneling. Consider the rectangular potential barrier of height  $V_0$  between  $-a/2$  and  $a/2$ . When  $E < V_0$  we know that the wave function in the classically forbidden region is a linear combination of  $\exp(-\kappa x)$  and  $\exp(+\kappa x)$  with  $\kappa$  real. Now I remember that the probability current density vanishes when the wave function is real. There is an incident current density and there is a transmitted current density because of tunneling. How can there be a current density for  $|x| > a/2$  without a current density in between? This quantum mechanics sure is weird." Provide a lucid explanation for this student with an explicit calculation. (5 points)

Misty, you had me scratching my head for a millisecond! The wave function in the forbidden region is

$$A e^{\kappa x} + B e^{-\kappa x}$$

as you say, however  $A$  and  $B$  are complex. Therefore,

$$\begin{aligned} \psi^* \psi' &= (A^* e^{\kappa x} + B^* e^{-\kappa x}) \kappa (A e^{\kappa x} - B e^{-\kappa x}) \\ &= \kappa (|A|^2 e^{2\kappa x} - |B|^2 e^{-2\kappa x}) + \kappa (AB^* - A^*B). \end{aligned} \quad (1)$$

Complex conjugation yields

$$\psi^{*\prime} \psi = \kappa (|A|^2 e^{2\kappa x} - |B|^2 e^{-2\kappa x}) + \kappa (A^*B - AB^*).$$

On subtraction, upon noting that  $AB^* - A^*B = 2i \Im[AB^*]$ , you obtain the probability current density,

$$j = \frac{2\hbar\kappa}{m} \Im[AB^*].$$

Since you are matching oscillatory wave functions with pure exponentials the coefficients will necessarily be complex. You can compute them explicitly if you have nothing else to do.

*So quantum mechanics is weird, but it ain't that weird.*

7.2) **Propagating waves in periodic potentials exhibit bands:**

The solution was sketched in class. Look at the infinite LC-ladder in the accompanying figure. I suggest that you simplify the boundary conditions by extending the ladder to  $-\infty$  and study the propagation of waves. Show that there is an allowed band of frequencies and find the upper bound on the propagating mode frequencies. (6 points)

**Discrete Transmission Line:** Label the charges on the capacitors by  $Q_n$  and the currents by  $I_n$  and let the current  $I_n$  flow between the capacitors with charges  $Q_{n-1}$  and  $Q_n$ . The equations of motion are given by Kirchoff's Voltage Law as you go around the loop containing two capacitors and an inductor

$$\frac{Q_{n-1}}{C} - L \frac{dI_n}{dt} - \frac{Q_n}{C} = 0$$

supplemented by charge conservation or Kirchoff's current Law at each node

$$I_n = I_{n+1} + \dot{Q}_n.$$

Differentiating the first equation and substituting into the second equation we obtain

$$L\ddot{I}_n + \frac{1}{C}(I_n - I_{n-1}) - \frac{1}{C}(I_{n-1} - I_{n-2}) = 0.$$

$$\text{Rearranging we obtain } \ddot{I}_n = \frac{1}{LC}[I_{n+1} - 2I_n + I_{n-1}].$$

We make the *ansatz*<sup>1</sup>  $I_n = A e^{in\theta - i\omega t}$ . Substituting we find  $\omega^2 = -\frac{1}{LC}[e^{i\theta} - 2 + e^{-i\theta}]$ , which simplifies to  $\omega^2 = \frac{2}{LC}[1 - \cos(\theta)] = \left[\frac{2}{\sqrt{LC}} \sin\left(\frac{\theta}{2}\right)\right]^2$  where the upper bound is given by  $\frac{2}{\sqrt{LC}}$ . Only frequencies in the band from zero to the upper bound propagate. This is a low-pass filter. If one writes  $\theta = ka$ , where  $a$  is the physical repeat distance, as  $k \rightarrow 0$  we find  $\omega = \frac{1}{\sqrt{\frac{L}{a}\frac{C}{a}}}k$  so that the speed of propagation depends on the inductance per unit length and capacitance per unit length.

**Exercise: Guess what happens if you exchange the inductances and capacitances and check it with an explicit calculation.**

**Continuum:** It is fairly easy to work out based on undergraduate level electromagnetic theory the theory of a transmission line (say, coaxial cable). All frequencies propagate with a dispersion relation  $\omega = vk$  where  $v = 1/\sqrt{L_0 C_0}$  where  $L_0$  is the inductance per unit length and  $C_0$  is the capacitance per unit length. Compare this with the discrete case. Note how free propagation is truncated to a band by the periodic structure. Using a cylindrical coaxial

---

<sup>1</sup>German word used in the sense of "an educated guess".

cable you can calculate the impedance of the coaxial cable and compare with the value of your TV cable.

**Find an analogous mechanical system with elementary components. (2 points)**

The system of springs and masses considered in class provides the analog. Try to make the analogy between mass and spring constant on the one hand and inductance and capacitance on the other precise.

*Consider  $2N$  (even) point particles in one dimension connected by springs. The spring constants alternate in strength with values  $\alpha_1$  and  $\alpha_2$ . You may assume that all the particles have the same mass  $m$  for algebraic simplicity. It is convenient to divide the  $2N$  particles into  $N$  cells each with two particles and denote the displacement of the two particles (from equilibrium) in the  $j^{\text{th}}$  cell by  $u_1(j)$  and  $u_2(j)$  respectively. Write down the classical equations of motion for the displacements from equilibrium making sure that even and odd sites have different behavior. Use periodic boundary conditions and find the allowed frequencies and wave vectors. How many bands do you obtain? When quantized these normal modes become **phonons**. (8 points)*

Imagine that the system is made of  $N$  cells of size  $a$  with two masses in each cell. The spring constant between the masses within a cell being denoted by  $\alpha_1$  and the spring constant between masses in different cells denoted by  $\alpha_2$ . Label the cells by an integer  $j$  and the two atoms in a cell by subscripts 1 and 2 and the corresponding displacements by  $u_1(j)$  and  $u_2(j)$ . The first atom in the  $j^{\text{th}}$  cell interacts with the second atom in the  $(j-1)^{\text{th}}$  cell with spring constant  $\alpha_2$  and with the second atom in the same cell with spring constant  $\alpha_1$ . This leads to

$$\ddot{u}_1(ja) = -\alpha_1[u_1(ja) - u_2(ja)] - \alpha_2[u_1(ja) - u_2((j-1)a)].$$

Similarly we obtain the equation of motion for the second atom in cell  $j$ :

$$\ddot{u}_2(ja) = -\alpha_1[u_2(ja) - u_1(ja)] - \alpha_2[u_2(ja) - u_1((j+1)a)].$$

Be absolutely clear about the equations of motion: it is freshman physics. Now how does one decouple these ODEs and find the normal modes?

**I will assume that  $\alpha_1 > \alpha_2$ .**

Each normal mode is characterized by a wave vector<sup>2</sup>  $k$  and a frequency  $\omega$ . So we make

---

<sup>2</sup>Note that the corresponding wavelength determines the periodicity in space of the displacements of the

the guess

$$u_1(ja) = A_1 e^{ik(ja) - i\omega t};$$

$$u_2(ja) = A_2 e^{ik(ja) - i\omega t}.$$

Since the two masses in a cell see different environments their amplitudes will not be the same. Substitute into the equations and note that factors of  $e^{-i\omega t}$  and  $e^{ik(ja)}$  cancel as usual. Substitution yields

$$[m\omega^2 - (\alpha_1 + \alpha_2)]A_1 + [\alpha_1 + \alpha_2 e^{-ika}]A_2 = 0 \quad (2)$$

$$[\alpha_1 + \alpha_2 e^{ika}]A_1 + [m\omega^2 - (\alpha_1 + \alpha_2)]A_2 = 0. \quad (3)$$

This pair of homogeneous equations will have a non-vanishing solution if the determinant of the coefficients vanishes. This yields

$$[m\omega^2 - (\alpha_1 + \alpha_2)]^2 = [\alpha_1 + \alpha_2 e^{-ika}] \times [\alpha_1 + \alpha_2 e^{ika}] = \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cos(ka).$$

Therefore we obtain two solutions

$$m\omega^2 = \alpha_1 + \alpha_2 \pm \sqrt{\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cos(ka)}.$$

One band extends from 0 to  $\sqrt{2\alpha_2/m}$ . The second band extends from  $\sqrt{2\alpha_2/m}$  to  $\sqrt{2(\alpha_1 + \alpha_2)/m}$ . See the figure below where I have chosen to rescale frequencies by  $\omega_0 = \sqrt{(\alpha_1 + \alpha_2)/m}$  and let  $\alpha_1 = 2\alpha_2$ .

**For those going into condensed matter physics:** Note the appearance of two bands with a band gap. The lower band with  $\omega \rightarrow c_s k$  for  $k \rightarrow 0$  (linear as  $k$  approaches zero representing sound waves) is called the **acoustic band**. The upper band is called the **optical band**. At  $k = 0$  the upper band frequency is given by  $m\omega^2 = 2(\alpha_1 + \alpha_2)$ . substituting into the amplitude equations one finds  $A_1 = -A_2$ . The two masses in a cell move out of phase with each other. If the two masses have positive and negative charges this corresponds to a time-varying dipole moment and this couples to electromagnetic waves. hence, the appellation "optic".

*7.3) Exercise 7.4.2, 7.4.5, and 7.4.6. Please read pages 203-212 and do these whether or not they are covered in class. They involve simple algebraic manipulations, albeit very useful. (6 points)*

These are easy and the answers are in the back of the book. So I will be lazy.

---

masses in that normal mode at a given time: Take a photograph with a high-speed camera and you can find the wavelength.

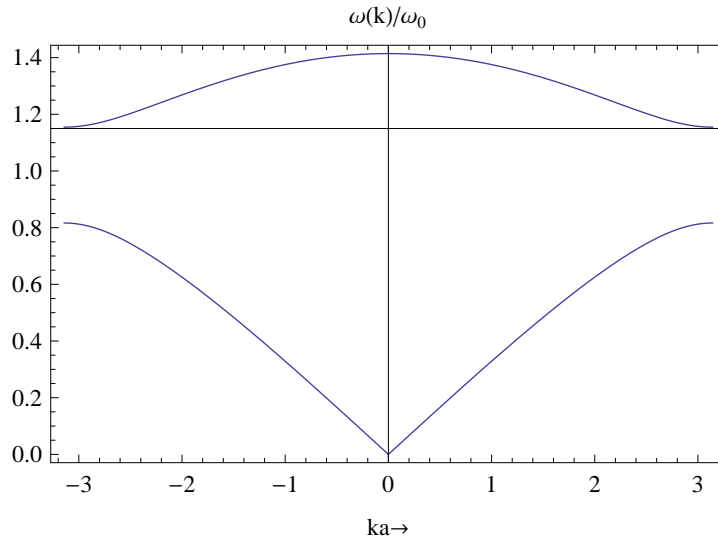


Figure 1: The dispersion relations  $\omega(k)/\omega_0$  vs  $k$  with  $\omega_0^2 = (\alpha_1 + \alpha_2)/m$  and  $\alpha_1 = 2\alpha_2$ .

7.4) Consider a perfectly flexible string with mass density  $\lambda$  under a tension  $T$ . Let  $\psi(x, t)$  be the transverse displacement at a position  $x$  at time  $t$ . You may assume that the equation of motion is

$$\frac{\partial^2 \psi(x, t)}{\partial t^2} = \frac{T}{\lambda} \frac{\partial^2 \psi(x, t)}{\partial x^2}.$$

It is good to review the derivation for your own edification but you need not submit it. Consider an infinite string and check that  $Ae^{ikx-i\omega t}$  yields a traveling wave solution and find the dispersion relation. Of course using the complex exponential is for mathematical convenience and there are no imaginary parts to a classical string.

Next consider an infinite string with a discontinuity in the mass density  $\lambda$  at  $x = 0$ :  $\lambda_1$  for  $x < 0$  and  $\lambda_2$  for  $x > 0$ . State clearly the boundary conditions satisfied by  $\psi(x, t)$  at  $x = 0$ . For a wave incident from the left with amplitude  $A_I$  (coefficient of  $e^{ikx-i\omega t}$ ) find the reflected and transmitted amplitudes denoted by  $A_R$  and  $A_T$ . Find  $A_T/A_I$  and  $A_R/A_I$  and compare your answers with those for the potential step in one-dimensional quantum mechanics.

If  $\lambda_2 > \lambda_1$  draw pictures for a localized wave incident from the left and the shapes of the transmitted and reflected wave forms. (10 points)

**Apologies:**  $\lambda$  is the mass per unit length, not a wavelength! Limitations of the length of the Greek and Latin alphabet. May be we should use Chinese characters.

As usual we take the wave forms to be

$$A_I e^{ik_1 x - i\omega t} + A_R e^{-ik_1 x - i\omega t}$$

for  $x < 0$  and for  $x > 0$  we have the transmitted wave:  $A_T e^{ik_2 x - i\omega t}$ . Note that we make  $\omega$  the same ensuring that the boundary conditions hold at all times. The dispersion relation can be read off from the wave equation by substitution of  $e^{ikx - i\omega t}$  to obtain  $\omega = vk$  where the speed  $v = \sqrt{T/\lambda}$ . So the wave vector in the two regions are determined by their respective densities and the common value of the tension  $T$ .

The boundary conditions when there is a discontinuity in the linear mass density of the string say at the origin are (a) continuity of the displacement of the string at all times which also yields continuity of the transverse velocity  $\partial_t \psi$  and (2) continuity of the transverse force on the string. If the latter condition is not satisfied then a point mass at the origin would suffer a finite force and an infinite acceleration. This implies that  $-[T\partial_x \psi(x, t)]_L = -[T\partial_x \psi(x, t)]_R$  which leads to a continuity of the slope. Note that this depends upon the tension being the same in both regions. Otherwise we get a slope discontinuity.

We obtain

$$A_I + A_R = A_T$$

$$k_1 A_I - k_1 A_R = k_2 A_T.$$

We find  $\boxed{\frac{A_R}{A_I} = \frac{k_1 - k_2}{k_1 + k_2}}$ . Since  $\omega/k = \sqrt{T/\lambda}$  we have  $k_1 \propto \sqrt{\lambda_1}$  and thus if  $\lambda_2 > \lambda_1$ , i.e., the wave goes from a lighter string to a heavier string the reflection coefficient is negative and the phase suffers a change of phase  $\pi$ , i.e., its sign changes. **A positive waveform gets reflected as an inverted waveform.** Show this in the figure!

We also obtain  $\boxed{\frac{A_T}{A_I} = \frac{2k_1}{k_1 + k_2}}$ . **Note that these answers are identical to those for reflection from a quantum mechanical potential step.**

Try re-writing the solution in terms of the characteristic impedances if you know basic circuit theory.