4.1. A system is described by the Hamiltonian H and we consider an observable denoted by S_y. The two operators are given by

$$H = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \quad \text{and} \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

(a) *What is the dimensionality of the Hilbert space to which a state vector describing the system belongs? Consider the state of the system described by* $|\psi_1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}$. *If one measures the observable denoted by* S_y *what are the possible results of the measurement and what are the corresponding probabilities? Explain precisely what it means to calculate probabilities.*

The Hilbert space is two-dimensional. The eigenvalues of S_y are $\pm\hbar/2$ and the eigenvectors are

$$|y+\rangle \;\rightarrow\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |y-\rangle \;\rightarrow\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad . \tag{1}$$

The possible results of the measurement are $\pm\hbar/2$ and the corresponding probabilities are

$$P_+ = |\langle y + |\psi_1\rangle|^2 = \frac{9}{10} \quad \text{and} \quad P_- = |\langle y - |\psi_1\rangle|^2 = \frac{1}{10}$$

If we prepare an ensemble of identically prepared systems described by $|\psi_1\rangle$ and measure S_y then 90% of the systems will yield a result $+\hbar/2$ as the number of systems becomes infinite.

(b) S_y *is measured on the ensemble described in (a) at* $t = 0$. *Those copies of the ensemble that yield the larger of the eigenvalues is selected. What is the state of the system immediately after the measurement? Denote this state by* $|\psi(t = 0)\rangle$.

$$|\psi(t = 0)\rangle = |y+\rangle .$$

(c) What is $|\psi(t)\rangle$ *for* $t > 0$? *Explain clearly the logic of your calculation.*

We expand the given state as a linear combination of the eigenstates of the Hamiltonian. The eigenstates of the Hamiltonian are

$$|+\epsilon\rangle \;\rightarrow\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\epsilon\rangle \;\rightarrow\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad . \tag{2}$$

We expand the given state as a linear combination of the eigenstates of the Hamiltonian.

$$|\psi(t=0)\rangle \; = \; c_+ \,|+\epsilon\rangle \; + \; c_- \,|-\epsilon\rangle$$

where

$$\begin{aligned} c_+ &= \langle +\epsilon|\psi(t=0)\rangle \; = \; (1+i)/2 \\ c_- &= \langle -\epsilon|\psi(t=0)\rangle \; = \; (1-i)/2 \\ |\psi(t)\rangle \; &= \; \frac{1+i}{2}\, e^{-i\epsilon t/\hbar}|+\epsilon\rangle \; + \; \frac{1-i}{2}\, e^{i\epsilon t/\hbar}|-\epsilon\rangle\,. \end{aligned}$$

You can write this in column vector form.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} [\cos(\epsilon t/\hbar) + \sin(\epsilon t/\hbar)] \\ i[\cos(\epsilon t/\hbar) - \sin(\epsilon t/\hbar)] \end{pmatrix}$$

(d) *At time t you measure S_y for an ensemble described by $|\psi(t)\rangle$ from part (c). Calculate the probability of measuring the the two eigenvalues and sketch them. Is there any connection between these results and the probabilities you found in (a)? If not why not?*

$$\begin{aligned} P_+(t) \; &= \; |\langle y+|\psi(t)\rangle|^2 \; = \; \cos^2(\epsilon t/\hbar) \\ P_-(t) \; &= \; |\langle y-|\psi(t)\rangle|^2 \; = \; \sin^2(\epsilon t/\hbar) \end{aligned}$$

These oscillate with a frequency $\Omega \; = \; 2\epsilon/\hbar$ as one would expect.
After the measurement all the information about the initial state is lost due to the collapse of the wave function. So this result is unrelated to the probabilities before measurement.

(e) *Find an explicit expression for the time evolution operator for this system as a 2×2 matrix. Use this to determine $|\psi_1(t)\rangle$ where $|\psi_1(t=0)\rangle \; = \; |\psi_1\rangle$ given in part (a).*
We note that

$$U(t) \; = \; e^{-iHt/\hbar} \; = \; e^{-i(\epsilon t/\hbar)\sigma_x} \; = \; \cos(\epsilon t/\hbar)\, I - i\sin(\epsilon t/\hbar)\,\sigma_x \; = \; \begin{pmatrix} \cos(\epsilon t/\hbar) & -i\sin(\epsilon t/\hbar) \\ -i\sin(\epsilon t/\hbar) & \cos(\epsilon t/\hbar) \end{pmatrix}.$$

You can use this representation and show that $|\psi(t)\rangle \; = \; U(t)\,|\psi(t=0)\rangle \; = \; U(t)|y+\rangle$ yields the answer given earlier in (c).

To obtain $|\psi_1(t)\rangle$ use the vector given in part (a) and compute $U(t)|\psi_1\rangle$.

(f) *Do you think an experimentalist can actually prepare a spin system in the state given in (a). If yes describe how one might do it.*
Yes. Find a linear combination of the Pauli matrices $\vec{\sigma}\cdot\hat{n}$ such that the given state is an eigenvector with eigenvalue 1. . A measurement of the spin along $\hat{n}$ will alow us to prepare the spin in this state. I will let you do the algebra,

(4.2) and (4.3) Exercises 4.2.2 *and* 4.2.3 *on page 139.*

We discussed **Exercise 4.2.2** in class. The key mathematical point to note that is that

$$\langle p|\psi\rangle \equiv \tilde{\psi}(p) \;=\; \int_{-\infty}^{\infty} dx\,\frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\psi(x)$$

and if $\psi(x)$ is real then $\tilde{\psi}(p) \;=\; \tilde{\psi}^*(-p)$. Thus the probability density for measuring p and $-p$ are the same. Multiplicative complex constants do not alter the argument.

Exercise 4.2.3
Let $\psi'(x) \;=\; e^{ip_0x/\hbar}\,\psi(x)$. The expectation value of the momentum in this state is $\langle\psi'|P|\psi'\rangle$ and in coordinate sapace we have

$$\int_{-\infty}^{\infty} dx\, e^{-ip_0x/\hbar}\psi^*(x)\left(p_0\, e^{ip_0x/\hbar}\psi(x) \;+\; e^{ip_0x/\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\right)$$

where I have carried out part of the differentiation. The first term yields p_0 since the wave function is normalized and the second term yields (upon canceling the exponential phase factor)

$$\int_{-\infty}^{\infty} dx\,\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x) \;=\; \langle P\rangle_\psi\,.$$