Problem Set 3 Solutions

3.1 (a) The mass of a baseball is roughly $0.14\,kg$.[1] Given the speed of $40m/s$[2] we obtain for the de Broglie wavelength

$$\lambda = \frac{h}{mv} = \frac{6.62 \times 10^{-34} J.s}{0.14kg \times 40m/s} \approx 1.2 \times 10^{-34} m\,.$$

Please do not just give the answer as a number without giving some number with which it can be compared. Recall that the size of the nucleus is of the order of a fermi ($10^{-15}m$) and so this is ludicrously small: no pitcher has to worry about the wave nature of the baseball.

(b) I will write the solution in a useful form. For *non-relativistic* electrons with kinetic energy T the momentum p is given by $p^2/(2m_e) = T$ where m_e is the mass of the electron. Thus $\lambda = \frac{h}{\sqrt{2m_eT}}$. Be careful to convert form eV to J if necessary with the conversion factor $1.6 \times 10^{-19} J/eV$. If $T = a\,eV$ where a is a numerical value we obtain

$$\lambda \approx \frac{1.23}{\sqrt{a}}\, nm\,.$$

This is a very useful formula: we just substitute the numerical value of the kinetic energy in eV, in our case $a = 10$ since we have already accounted for the units. We obtain $\lambda \approx 0.39\, nm$.

(1c) We have for the photon $E_\gamma = \frac{hc}{\lambda_\gamma}$ where $hc \approx 1240\, eV.nm$. For the electron

$$T_e = \frac{h^2}{\lambda_e^2}\frac{1}{2m_e} = \frac{h^2c^2}{\lambda_e^2}\frac{1}{2m_ec^2}\,.$$

Equating the energies with $\lambda_e = \lambda_\gamma = \lambda^*$ we obtain

$$\lambda^* = \frac{hc}{2m_ec^2} = 1240eV.nm/(2 \times 0.511MeV) \approx 1.21\, pm\,.$$

Since we know λ^* we know the corresponding energy $hc/\lambda^* = 2m_ec^2$. The kinetic energy is twice the rest mass. The non-relativistic approximation is not very good. This calculation also shows that when the wavelengths of the photon and the massive

[1] It cannot be $10g$: it is worth remembering that a nickel weighs $5g$. It cannot be $1kg$: not even Barry Bonds on steroids could it hit out of the ballpark. If you take the geometric mean you get $100g$ and that is acceptable.

[2] $40m/s$ is roughly $90mph$. Reasonable, I'm just watching Sabathia pitching for the Yankees.

particle are equal the kinetic energy is twice the rest energy. m_e could have been any mass. The specific particle does not matter. As the energy increases far beyond the rest mass the electron acts as if it is massless and to a very good approximation both behave the same way.

3.2 The (Bohr) radius of the hydrogen atom is $0.53 \times 10^{-10} m$. The radius of a large atom is roughly twice this radius. The radius of the earth is approximately $6400 km$. So if, as claimed,

$$\frac{r_{earth}}{r_{apple}} = \frac{r_{apple}}{r_{atom}}$$

we can "predict" the radius of the apple to be

$$r_{apple} = \sqrt{10^{-10} \times 6400 \times 10^3} m \approx 2.5 cm \,.$$

That is an inch. Close enough for (state) government work!

3.3 Note that the work function is a sensitive function of the impurities on the surface. Recall that if the frequency of the incident photon is less than the cutoff frequency no electrons are ejected from Sodium in this case. The work function is defined as the minimum energy required to take an electron from the interior of the metal to just outside. Therefore, we have

$$h\nu_{cutoff} = W \,.$$

One can use $h = 4.14 \times 10^{-15} eV.s$ and the given value of $4.4 \times 10^{14}\, s^{-1}$ for the cutoff frequency to obtain $W \approx 1.82\, eV$. The binding energy of the hydrogen atom (a number to remember) is $13.6 eV$.

For (i) $h\nu < W$ there is no current as a function of the voltage. However, if the voltage is made very large and positive (depending on the geometry) it can yank some electrons off the cathode and lead to a current. This is called field emission and we will ignore this.

(ii) $h\nu > W$: The current is zero for $V < V_{stopping}$ and then increases rapidly over a very short range of voltages and saturates for positive V: all the emitted electrons at that incident intensity are being swept up by the collector plate.

(iii) As the intensity increases at the same frequency of incident light, the stopping voltage is unaltered. However, more photons are incident (higher intensity) and more photoelectrons will be ejected and the saturation current increases roughly linearly with the intensity. So over some range you would expect the current to double as the intensity doubles. Note that not every photon ejects an electron.

(iv) Note that as the frequency of the photon is increased electrons more strongly bound in the metal can be ejected. Therefore many more electrons will be ejected and the saturation current can become much larger. This effect is non-linear. In addition the stopping voltage becomes much higher since the maximum kinetic energy of the liberated electrons is higher. Recall that

$$eV_{stop} = h\nu - W \,.$$

So when we double ν, the stopping voltage increases by an additional $h\nu/e$.

Instead of giving you figures I will direct you to a very nice site. I urge you to look at the graphs.

`http://phet.colorado.edu/simulations/sims.php?sim=Photoelectric_Effect`

3.4) Shankar **1.10.4**

This is just a matter of doing integrals. One of the integrals is

$$\frac{2}{L}\int_0^{L/2} dx\, \frac{2xh}{L}\, \sin\left(\frac{m\pi x}{L}\right) = \frac{4h}{\pi^2}\int_0^{\pi/2} dy\, y\, \sin(my)$$

where we have changed variables to $y = \pi x/L$. The other integral from $L/2$ to L is similarly

$$\frac{4h}{\pi^2}\int_{\pi/2}^{\pi} dy\ (\pi - y)\ \sin(my)\,.$$

Changing variables again to $z = \pi - y$ we have

$$-\frac{4h}{\pi^2}\int_0^{\pi/2} dz\, z\, \cos(m\pi)\, \sin(mz)\,.$$

Adding we obtain

$$[1 - \cos(m\pi)]\,\frac{4h}{\pi^2}\int_0^{\pi/2} dy\, y\, \sin(my)\,.$$

We only need the integral for odd m since for even m the factor in front vanishes. Doing this integral either by parts or using Mathematica yields the result given in the text.

$$\int_0^{\pi/2} dy\, y\, \sin(my) \;=\; -\frac{y}{m}\cos(my)\Big|_0^{\pi/2} + \frac{1}{m}\int_0^{\pi/2} dy\, \cos(my)\,.$$

The first term vanishes for odd m since the cosine vanishes for odd multiples of $\pi/2$. The remaining integral along with the factors yields

$$\frac{8h}{m^2\pi^2}\,\sin(m\pi/2)\,.$$

3.5) Shankar **4.2.1**

1) L_z is in the diagonal representation and so we can read off the eigenvalues, 1, 0, and -1. Please do not write any equations. The eigenvectors are obvious.

2) It is easy to evaluate the required quantities:

$$\langle 1|L_x|1\rangle \;=\; (1,\,0,\,0)\begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0\,.$$

$$\langle 1|L_x^2|1\rangle \;=\; (1,\,0,\,0)\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\,.$$

Therefore, $\Delta L_x \;=\; 1/\sqrt{2}$.

3) Simple Linear Algebra: The eigenvalues of L_x are 1, 0, and -1 and the eigenvectors are

$$|L_x = 1\rangle \;=\; \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix},\quad |L_x = 0\rangle \;=\; \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix},\quad |L_x = -1\rangle \;=\; \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix},$$

4) The possible outcomes of the measurement are 1, 0, and -1 by the postulates. The probabilities of obtaining these in the state $|L_z = -1\rangle$ are obtained by computing the appropriate inner products and taking their absolute value squared, i.e., $|\langle L_x = 1|L_z = -1\rangle|^2$, $|\langle L_x = 0|L_z = -1\rangle|^2$, and $|\langle L_x = -1|L_z = -1\rangle|^2$. These are respectively 1/2, 1/4, and 1/2. By inspection it should be clear to you that the inner

product is the third element of each of eigenvectors of L_x.

5) **Please read page 124 of the text carefully and the answer follows.** Clearly the state after measuring L_z^2 given that we obtain a value 1 is $P_1|\psi\rangle/|P_1|\psi\rangle|$ where P_1 is the projection operator onto the subspace associated with the eigenvector 1 and the denominator is just the normalization factor. Clearly the projection operator P_1 projects out the first and third components. $L_z^2 = 1$ corresponds to $L_z = \pm 1$ and the projection operator P_1 corresponds to $|1\rangle\langle 1| + |-1\rangle\langle -1|$. Thus we are left with

$$P_1|\psi\rangle \iff \begin{pmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} .$$

We normalize this to find the state after measuring L_z^2 for the given $|\psi\rangle$ to be

$$\begin{pmatrix} 1/\sqrt{3} \\ 0 \\ \sqrt{2}/\sqrt{3} \end{pmatrix} .$$

The probability for this result is 3/4. One way is to compute the probability of measuring $L_z^2 = 0$ or $L_z = 0$ in the state $|\psi\rangle$ which is 1/4, the square of the second element in the column vector.

The rest of the problem is ambiguous. If I interpret it to mean that L_z is measured after the L_z^2 measurement then the probabilities of obtaining 1, 0, and -1 are respectively1/3, 0, and 2/3 respectively as can be read off.

6) The first part is trivial. If $|a|^2 = 1/4$ then in general, $a = \frac{1}{2}e^{i\delta_1}$ for real δ_1 from baby complex variable theory. The problem emphasizes the fact that while the overall phase factor is irrelevant the relative phases do indeed matter. To find the probability of obtaining $L_x = 0$ we first calculate the following matrix element:

$$\langle L_x = 0|\psi\rangle = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \begin{pmatrix} e^{i\delta_1}/2 \\ e^{i\delta_2}/\sqrt{2} \\ e^{i\delta_3}/2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \left(e^{i\delta_1} - e^{i\delta_3}\right) .$$

Note that (please check this and be able to do the following quickly)

$$\left|e^{i\delta_1} - e^{i\delta_3}\right|^2 = 2\left[1 - \cos(\delta_1 - \delta_3)\right]$$

or in other words the answer for the probability depends on the relative phases. If the overall phase is altered, i.e., $\delta_i \to \delta_i + \phi$ for $i = 1, 2, 3$ then the result is unaltered in this case and in general.