2.1) (7 points) Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 5 & 12 \\ 4 & 3 \end{pmatrix} .$$

Are the eigenvectors orthogonal? Find a row vector that is orthogonal to the eigenvector corresponding to the largest eigenvalue. Can you find any interesting property that this row vector obeys? If you cannot find one just say so.

The eigenvalues are 11 and -3 with eigenvectors

$$5^{-1/2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad 13^{-1/2} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad \text{respectively.}$$

It is evident that the dot product does not vanish. It is easy to see that

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

is orthogonal to the first eigenvector. Any multiple will do as well.
What interesting property does it satisfy? The corresponding row (bra) is a left eigenvector

$$(1, \; -2) \begin{pmatrix} 5 & 12 \\ 4 & 3 \end{pmatrix} = -3\,(1, \; -2)$$

with eigenvalue -3. **If you are theorist wannabe can you find the matrix that can be used to diagonalize this matrix by a similarity transformation?**

2.2)(10 points) Consider a matrix which is perturbed slightly from the matrix Ω considered in the text on page 38, *Example 1.8.5*

$$\Omega_\epsilon = \begin{pmatrix} 1 & 0 & 1-\epsilon \\ 0 & 2 & 0 \\ 1-\epsilon & 0 & 1 \end{pmatrix} .$$

Find its eigenvalues. Are they distinct? (What do you obtain when $\epsilon \to 0$)? Do you expect Ω_ϵ to possess orthogonal eigenvectors? Why? Find the eigenvectors of Ω_ϵ and determine the limit $\epsilon \to 0$.
What exactly does this calculation show? Please provide a brief but precise answer.

Construct the unitary matrix that diagonalizes ω_ϵ. State exactly what this means. You do not have to carry out any matrix multiplications.

Note that Ω_ϵ is symmetric (hence, Hermitian) it has real eigenvalues. If the eigenvalues are distinct the normalized eigenvectors form an orthonormal set. The following Mathematica commands

```
M = {{1, 0, 1 - \[Epsilon]}, {0, 2, 0}, {1 - \[Epsilon], 0, 1}}
Eigensystem[M]
```

yield the eigenvalues 2, $2-\epsilon$, and ϵ and the **unnormalized** eigenvectors. . As $\epsilon \to 0$ these reduce the values obtained by Shankar. The coresponding normalized eigenvectors are

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

respectively. These form an orthonormal basis. As $\epsilon \to 0$ we recover Ω in Shankar and automatically obtain an orthonormal basis for the matrix with degenerate eigenvalues!

If the eigenvalues are degenerate consider Ω as the limit of nearby Hermitian matrices Ω_ϵ with distinct eigenvalues, For every $\epsilon > 0$ there are n mutually orthogonal eigenvectors. As $\epsilon \to 0$, the eigenvalues and eigenvectors vary smoothly. Two (or more) roots of the characteristic equation may coalesce leading to degenerate eigenvalues. The angle between the corresponding eigenvectors will not jump suddenly from $\pi/2$ to zero leading to a lower-dimensional space. Thus one will continue to have n linearly independent eigenvectors even in the degenerate case from which an orthonormal basis can be constructed.
The unitary matrix is constructed by assembling the **normalized** eigenvectors to form its columns:[1]

$$U = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

What we mean is that $U^{-1}\Omega_\epsilon U$ is diagonal with the eigenvalues as the diagonal elements.

[1] Once more with emphasis, the columns of a unitary matrix ate orthonormal vectors.

2.3)(6 points) *Exercise 1.8.5* on page 42.

The columns form an orthonormal vector as is evident by inspection. Or compute $\Omega^T\Omega$ and find I. We have an orthogonal matrix.

The characteristic equation is

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0 \quad \Rightarrow \quad \lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}\,.$$

The corresponding eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix},.$$

$$\frac{1}{\sqrt{2}}(1,\, i)\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = 0\,.$$

Remember that U is obtained by assembling the orthonormal eigenvectors as columns:

$$U = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}.$$

Clearly, ΩU is given by (you should see this without explicit multiplication)

$$\begin{pmatrix}\frac{e^{i\theta}}{\sqrt{2}} & \frac{e^{-i\theta}}{\sqrt{2}}\\ \frac{ie^{i\theta}}{\sqrt{2}} & -\frac{ie^{-i\theta}}{\sqrt{2}}\end{pmatrix}.$$

It is straightforward to pre-multiply by

$$U^\dagger = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}$$

and obtain

$$U^\dagger\Omega U = \begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}.$$

2.4) (8 points) *Exercise 1.8.12* on page 54.

This is a simple plug-and-chug problem. We know the equation of motion

$$|\ddot{x}\rangle = \Omega|x(t)\rangle\,.$$

We write, as instructed, the solution in the form

$$|x(t)\rangle = U(t)|x(0)\rangle$$

and substitute into the equation of motion. Differentiating this equation twice with respect to time and equating it to $\Omega|x(t)\rangle$ we have

$$\ddot{U}\,|x(0)\rangle \;=\; |\ddot{x}\rangle \;=\; \Omega|x(t)\rangle \;=\; \Omega U(t)|x(0)\rangle\,.$$

Therefore, we obtain (since this is true for all initial conditions)

$$\ddot{U}(t) \;=\; \Omega U(t)$$

which is a matrix differential equation. One boundary condition is clearly $U(t=0) = I$. Since the velocities vanish $\dot{U}(t=0) \;=\; 0$.

Just to be excruciatingly explicit we have

$$\begin{pmatrix} \ddot{U}_{11}(t) & \ddot{U}_{12}(t) \\ \ddot{U}_{21}(t) & \ddot{U}_{22}(t) \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix}.$$

These are 4 coupled differential equations. Fortunately, since you are told that $[U,\,\Omega] \;=\; 0$ we can convert this matrix differential equation into a pair of decoupled ordinary differential equations. Go to the $\{|I\rangle,\,|II\rangle\}$ basis that diagonalizes Ω and U as well. In this basis we have

$$\begin{pmatrix} \ddot{U}_{I,I}(t) & 0 \\ 0 & \ddot{U}_{II,II}(t) \end{pmatrix} = \begin{pmatrix} -\omega_I^2 & 0 \\ 0 & -\omega_{II}^2 \end{pmatrix} \begin{pmatrix} U_{I,I}(t) & 0 \\ 0 & U_{II,II}(t) \end{pmatrix}.$$

We have two decoupled differential equations: thus $U_{I,I}(t) \;=\; \cos\omega_I t$ using the two initial conditions. The sine term vanishes since the velocity is zero at $t=0$. We obtain $U_{II,II}$ similarly leading to Equation (1.8.43) on page 53.

2.5) (10 points) *Exercise 1.10.4* on page 73.

Postponed

2.0) You need not submit these: *Exercises 1.10.1-1.10.3* on page 63. These results will be needed in the rest of the course. You should remember them.