PHYSICS 827 HW1

Due Wednesday 09/30 by 4:59PM in Mr. Nick Harmon's mailbox

1.1 Shankar Exercise 1.3.4 (p17) (10 points)

Solution: We have

$$\begin{aligned} |V+W|^2 &= \langle V+W|V+W\rangle && (1)\\ &= \langle V|V\rangle + \langle W|W\rangle + \langle V|W\rangle + \langle W|V\rangle && (2)\\ &= \langle V|V\rangle + \langle W|W\rangle + 2\,\Re[\langle V|W\rangle] && (3)\end{aligned}$$

Now we use the fact

$$|\langle V|W\rangle|^2 = (\Re[\langle V|W\rangle])^2 + (\Im[\langle V|W\rangle])^2 \quad (4)$$

and obtain

$$\Re[\langle V|W\rangle] \le |\langle V|W\rangle|\,. \quad (5)$$

Note in passing that equality obtains when $\Im\langle V|W\rangle = 0$ a result used in the second part.

$$\begin{aligned} \text{Thus,}\ |V+W|^2 &\le \langle V|V\rangle + 2|\langle V|W\rangle| + \langle W|W\rangle && (6)\\ &\le |V|^2 + 2|V|\,|W| + |W|^2 \quad \text{by Schwarz inequality}\\ &= (|V| + |W|)^2\,. && (7)\end{aligned}$$

(Recall that the Schwarz inequality states that $|\langle V|W\rangle| \le |V|\,|W|\,$.) Thus we have shown that $\boxed{|V+W| \le |V| + |W|}$.

When is equality achieved? Obvious for real vectors from elementary geometry. The two vectors should be parallel to each other, i.e., proportional with a positive proportionality constant. The proof in our case is straightforward. I will just emphasize the idea. There are two places in the proof above we used inequalities where we should obtain equalities. From the first in Equation (5) we have as noted the condition

$$\Re[\langle V|W\rangle] = |\langle V|W\rangle|$$

which implies that $\langle V|W\rangle = \langle W|V\rangle$ is real and positive. The other point is when we used Schwarz's inequality. For it to be an equality

$$|\langle V|W\rangle| = |V|\,|W|\,.$$

This implies that $|W\rangle = a|V\rangle$ where a is real and positive.[1] The converse is obviously true.

1.2 Shankar Exercise 1.6.2 (p27) (5 points)

We recall that the determinant of the transpose of a matrix is the determinant of the matrix. The determinant of the complex conjugate of a matrix is the complex conjugate of the determinant. Both of these should be obvious results. Thus we have

$$(\det U)^* = \det U^\dagger .$$

We know that $UU^\dagger = I$. Using the fact that the determinant of the product is the product of the determinants we have

$$\det(U)\det(U^\dagger) = \det U \times \det U^\dagger = 1.$$

Using the earlier result

$$|\det U|^2 = 1 \quad \Rightarrow \quad \det U = e^{i\alpha} \quad \alpha \in \Re.$$

Another solution: The determinant is the product of the eigenvalues. (One way to see this is in the representation in which U is diagonal.) Since the eigenvalues of a unitary matrix are each of unit modulus the result follows.

1.3 Shankar Exercise 1.6.4 (p29) (5 points)

We determine whether the given operators are Hermitian:

$$(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger = \Lambda\Omega$$

Thus it is not Hermitian unless the two operators commute.

$$(\Omega\Lambda + \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger = \Lambda\Omega + \Omega\Lambda .$$

[1] If you are worried that

$$|W\rangle = a|V\rangle + c|Z\rangle$$

where $|Z\rangle$ is perpendicular to $|V\rangle$ This possibility is easily eliminated.

Therefore, (2) is Hermitian.

$$[\Omega,\, \Lambda]^\dagger \;=\; -[\Lambda,\, \Omega]\,.$$

Therefore, (3) is not Hermitian.
It is easy to verify that (4) is Hermitian because of the change of sign of i upon conjugation.

1.4 (10 points) Consider a three-dimensional Hilbert space spanned by the orthonormal basis $|1\rangle$, $|2\rangle$, and $|3\rangle$. The kets $|\psi\rangle$ and $|\phi\rangle$ are given by

$$|\psi\rangle \;=\; a|1\rangle \;+\; b|2\rangle \;+\; a|3\rangle \quad \textit{and} \quad |\phi\rangle \;=\; b|1\rangle \;-\; a|2\rangle$$

where a and b are complex constants.

(a) Determine $\langle\psi|$, $\langle\phi|$, $\langle\psi|\phi\rangle$ and $\langle\phi|\psi\rangle$. Under what conditions are $|\psi\rangle$ and $|\phi\rangle$ orthogonal?

$$\begin{aligned}
\langle\psi| \;&=\; \langle 1|a^* \;+\; \langle 2|b^* \;+\; \langle 3|a^* \\
\langle\phi| \;&=\; \langle 1|b^* \;-\; \langle 2|a^* \\
\langle\psi|\phi\rangle \;&=\; a^*b \;-\; ab^* \\
\langle\phi|\psi\rangle \;&=\; ab^* \;-\; a^*b\,.
\end{aligned}$$

For $|\psi\rangle$ and $|\phi\rangle$ to be orthogonal we need a^*b to be real. Clearly, this implies that the phase of the complex numbers of a and b are the same: $a \;=\; |a|e^{i\theta}$ and $b \;=\; |b|e^{i\theta}$.

(b) Express $|\psi\rangle\langle\phi|$ as a 3×3 matrix, i.e., determine all the matrix elements in the given orthonormal basis.
We know that $\langle i|\psi\rangle\langle\phi|j\rangle$ is the ij^{th} matrix element. We have

$$\begin{pmatrix} ab^* & -aa^* & 0 \\ bb^* & -ba^* & 0 \\ ab^* & -aa^* & 0 \end{pmatrix}.$$

(c) Consider the operator $P \;\equiv\; |\psi\rangle\langle\psi| \;+\; |\phi\rangle\langle\phi|$. Is P Hermitian? What is P^2? Give a simple argument (no explicit calculations) to show that P has a zero eigenvalue.
Clearly $P^\dagger \;=\; |\psi\rangle\langle\psi| \;+\; |\phi\rangle\langle\phi| \;=\; P$. We also have

$$P^2 \;=\; A|\psi\rangle\langle\psi| \;+\; B|\phi\rangle\langle\phi| \;+\; C|\psi\rangle\langle\phi| \;+\; C^*|\phi\rangle\langle\psi|$$

where $A = \langle\psi|\psi\rangle = 2|a|^2 + |b|^2$, $B = \langle\phi|\phi\rangle = |a|^2 + |b|^2$, and $C = \langle\psi|\phi\rangle = a^*b - ab^*$.

$|\psi\rangle$ and $|\phi\rangle$ correspond to two linearly independent vectors in three-dimensional space. They define a plane. Thus P acting on a vector normal to the plane, i.e., to $|\psi\rangle$ and $|\phi\rangle$ is annihilated by P yielding a zero eigenvalue for P.