The operator L_x is given in matrix form as

$$\hbar \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix} .$$

The eigenvalues of L_x are $\hbar$, 0, and $-\hbar$ and the eigenvectors are given respectively by

$$|1\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad |-1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad .$$

The momentum eigenfunction in one dimension with eigenvalue p is given by

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} .$$

$$\int_{-\infty}^{\infty} dk \frac{k^2}{(k^2+\alpha^2)^2} = \frac{\pi}{2\alpha} .$$

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha|x|}\, e^{-ikx} = \frac{2\alpha}{\alpha^2+k^2} .$$

$$\int_{0}^{\infty} dx\, e^{-\beta x} = \frac{1}{\beta}$$

I. *The Hamiltonian of a system is given by*

$$H = \hbar\omega \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

You will answer a series of questions about this system. Different parts may or may not be related and are indicated clearly.

(a) (4 points) *If the state of the system is described by* $|\psi_1\rangle \rightarrow (1/10) \begin{pmatrix} 7 \\ \sqrt{2}i \\ 7 \end{pmatrix}$ *what are the possible results of an energy measurement and the corresponding probabilities?*

The energy measurement can only yield the eigenvalues of the Hamiltonian, to wit, $2\hbar\omega$, 0, and $-2\hbar\omega$ with probabilities 49/100, 1/50 and 49/100 respectively.

(b)(4 points) Given the state $|\psi_1\rangle$ you measure the observable L_x, the operator corresponding to which is defined on the first page. What is the probability of obtaining the value $+\hbar$? What is the state of the system immediately after the measurement? Denote this state by $|\psi(0)\rangle$.

The probability is $|\langle L_x = \hbar|\psi_1\rangle|^2$ Since we have

$$\langle 1|\psi_1\rangle = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)(1/10)\begin{pmatrix} 7 \\ \sqrt{2}i \\ 7 \end{pmatrix} = \frac{1}{10}\left[(7/2) + i + (7/2)\right] = \frac{7+i}{10}$$

the probability is $\boxed{1/2}$. Immediately after the measurement the system is in the state denoted by

$$|1\rangle \rightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

(c)*(4 points) Does the state of the system immediately after the measurement in (b), $|\psi(0)\rangle$, have a definite energy eigenvalue? If so what is it? If not explain succinctly why not.*

No, the system is not an energy eigenstate; it can be found in the three energy states with probabilities 1/4, 1/2, and 1/4 as can be seen explicitly. The formal reason is that $[H, L_x] \neq 0$.

(d) (5 points) Given $|\psi(0)\rangle$ from part (b) find the state of the system at time t denoted by $|\psi(t)\rangle$.

Since H is diagonal its energy eigenfunctions are obvious and the given state can be expanded trivially in terms of these:

$$|\psi(0)\rangle = \frac{1}{2}|2\hbar\omega\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-2\hbar\omega\rangle.$$

$$|\psi(0)\rangle \rightarrow \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using the standard recipe we obtain

$$|\psi(t)\rangle \rightarrow \frac{e^{2i\omega t}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{e^{-2i\omega t}}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(e)*(6 points) Suppose we measure L_x on the system described by $|\psi(t)\rangle$ from (d). Determine the probability of finding the value $+\hbar$? Make a careful sketch of the probability as a*

function of time; specify the time at which the probability returns to the initial value.
The probability of obtaining $L_x = 1$ in $|\psi(t)\rangle$ is $|\langle 1|\psi(t)\rangle|^2$. We have

$$\langle 1|\psi(t)\rangle \;=\; \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right) \begin{pmatrix} \frac{e^{2i\omega t}}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{e^{-2i\omega t}}{2} \end{pmatrix} \;=\; \frac{1}{2}(1 \,+\, \cos 2\omega t)\,.$$

So the probability is $\boxed{(1/4)(1+\cos 2\omega t)^2 \;=\; \cos^4 \omega t}$. Obviously at $t = 0$ this probability is unity and it returns to its original value when $T = \pi/\omega$.

(f) (5 points) Compute the average value of L_x in the state at time t computed in (d).

You can obtain this by computing $\langle\psi(t)|L_x|\psi(t)\rangle$ directly.

Here is another way: The average value of L_x is the sum of $\hbar$ times the probability of obtaining $+\hbar$ and $-\hbar$ times the probability of obtaining $-\hbar$.
The probability of obtaining $L_x = -\hbar$ is computed from

$$\langle -1|\psi(t)\rangle \;=\; \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \begin{pmatrix} \frac{e^{2i\omega t}}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{e^{-2i\omega t}}{2} \end{pmatrix} \;=\; \frac{1}{2}(-1 \,+\, \cos 2\omega t)\,.$$

The probability is $\boxed{(1/4)(1-\cos 2\omega t)^2 \;=\; \sin^4 \omega t}$. Similarly the probability of obtaining $L_x = 0$ is can be computed but we do not need it. (Why not?)

$$\langle L_x\rangle(t) \;=\; \hbar\,(\cos^4 \omega t \;-\; \sin^4 \omega t) \;=\; \hbar\,\cos 2\omega t\,.$$

II. (a) (4 points) *Which of the following two operators will* necessarily *have real eigenvalues and why?*

$$|\psi\rangle\langle\phi| \,+\, |\phi\rangle\langle\psi|$$

$$|\psi\rangle\langle\phi| \,-\, |\phi\rangle\langle\psi|$$

Denote the first operator by Ω_1. We have

$$\Omega_1^\dagger \;=\; |\phi\rangle\langle\psi| \,+\, |\psi\rangle\langle\phi| \;=\; \Omega_1$$

Therefore. Ω_1 is Hermitian and its eigenvalues are real.
Denote the second operator by Ω_2. We have

$$\Omega_2^\dagger \;=\; |\phi\rangle\langle\psi| \,-\, |\psi\rangle\langle\phi| \;=\; -\Omega_2$$

Therefore. Ω_2 is **not** Hermitian and its eigenvalues will not necessarily be real. (Indeed, in this case they are pure imaginary.)

(b) *(8 points)* Λ *is a (linear) Hermitian operator and* $|\lambda\rangle$ *are eigenvectors for different values of* λ*:*

$$\Lambda|\lambda\rangle \;=\; \lambda|\lambda\rangle\,.$$

Ω *is a linear operator that obeys the commutation relation*

$$[\Omega,\,\Lambda] \;=\; \Omega\,.$$

(i) Show that $\Omega|\lambda\rangle$ *is an eigenvector of* Λ *and find the corresponding eigenvalue.*
(ii) Is $\Omega^\dagger|\lambda\rangle$ *an eigenvector of* Λ*? Check this explicitly and if it is calculate the corresponding eigenvalue.*
We wish to check that $\Omega|\lambda\rangle$ is an eigenvector of Λ. So we compute $\Lambda\Omega|\lambda\rangle$ by using the commutation relation: $\Lambda\Omega \;=\; \Omega\Lambda \;-\; \Omega$. We have

$$\Lambda\Omega|\lambda\rangle \;=\; \Omega\Lambda|\lambda\rangle \;-\; \Omega|\lambda\rangle \;=\; (\lambda-1)\,\Omega|\lambda\rangle$$

proving the assertion and identifying the eigenvalue as $\lambda - 1$.

We can use a similar stratagem for $\Omega^\dagger$ if we had the commutation relation for $\Omega^\dagger$ with Λ. So we take the adjoint of the given commutation relation to obtain

$$\Lambda\Omega^\dagger \;-\; \Omega^\dagger\Lambda \;=\; \Omega^\dagger\,.$$

So we calculate:

$$\Lambda\Omega^\dagger|\lambda\rangle \;=\; \Omega^\dagger\Lambda|\lambda\rangle \;+\; \Omega^\dagger|\lambda\rangle \;=\; (\lambda+1)\,\Omega^\dagger|\lambda\rangle$$

verifying that $\Omega^\dagger|\lambda\rangle$ is indeed an eigenvector of Λ and identifying the eigenvalue as $\lambda + 1$.

III. Your precocious pre-teen niece has solved for the ground state of a one-dimensional delta function potential and finds that the wave function[1] *is given by*

$$\psi(x) \;=\; \sqrt{\alpha}\,e^{-\alpha|x|}$$

where α *is real and positive. She asks, "If I measure the momentum in this state*
(a) (6 points) what is the probability of obtaining a value between p_0 *and* $p_0 + dp$*? and*
(b) (4 points) What is the width of the momentum distribution?" How does the width depend on α*? If you do not know how to calculate it guess the answer with an explanation.*
Please explain the logic of your calculation very briefly. Some useful integrals are provided

[1] "Of course, I normalized it, silly."

on the first page.

The logic is discussed with great clarity in Shankar on page 136. The procedure is obvious since the probability density is given by $|\langle p|\psi\rangle|^2$. So we compute

$$\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx \langle p|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx \, \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \sqrt{\alpha}\, e^{-\alpha|x|} \, .$$

The integral can be done using the result on the front page and we obtain $\frac{1}{\sqrt{2\pi\hbar}} \frac{2\alpha^{3/2}}{\alpha^2+(p/\hbar)^2}$. Thus the required probability is

$$\frac{1}{2\pi\hbar} \frac{4\alpha^3}{(\alpha^2 + (p_0/\hbar)^2)^2} \, dp \, .$$

The mean value for the above probability density clearly vanishes. (Why?) We can find $\langle p^2\rangle$:

$$\int_{-\infty}^{\infty} dp \, p^2 \, \frac{1}{2\pi\hbar} \frac{4\alpha^3}{(\alpha^2 + (p/\hbar)^2)^2} = \hbar^2 \int_{-\infty}^{\infty} dk \, k^2 \, \frac{1}{2\pi} \frac{4\alpha^3}{(\alpha^2 + k^2)^2} = \hbar^2\alpha^2 \, .$$

We have used the integral given on the first page after substituting $p = \hbar k$. Thus the width is $\boxed{\hbar\alpha}$. We can guess this since the width of the real space wave function is $1/\alpha$, in momentum space the width will be proportional to its inverse.