Scalar and vector potentials: We introduced the scalar potential \( V(\vec{r}, t) \) and \( \vec{A}(\vec{r}, t) \) so that they are solutions to two of Maxwell’s equations. The electric and magnetic fields can be obtained using

\[
\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \tag{1.1}
\]

\[
\vec{B} = \vec{\nabla} \times \vec{A} \tag{1.2}
\]

Clearly \( \vec{\nabla} \cdot \vec{B} \) vanishes since \( \vec{B} \) is the curl of a vector field. Taking the curl of the first equation we find that it yields Faraday’s Law. (Please verify this!) The other two equations which contain the sources will be used to obtain partial differential equations for \( V \) and \( \vec{A} \) (with sources). Armed with the solution of the resultant wave equations with sources we can obtain the electric and magnetic fields as a function of \( \vec{r} \) and \( t \) using the equations (1.1) and (1.2).

Gauge transformations: As an aside we note that the potentials are not unique. The fields and not the potentials are the physically measurable quantities\(^1\). Hence, any transformation of the potentials which leaves the fields unchanged (invariant), known as gauge transformation, does not change the results of the theory. Given below are the gauge transformations that leave the fields invariant:

\[
\vec{A} \rightarrow \vec{A}'(\vec{r}, t) = \vec{A} + \vec{\nabla} \chi(\vec{r}, t) \tag{1.3}
\]

\[
V \rightarrow V'(\vec{r}, t) = V(\vec{r}, t) - \frac{\partial \chi(\vec{r}, t)}{\partial t} \tag{1.4}
\]

where \( \chi(\vec{r}, t) \) is a (twice-)differentiable function of its arguments. Please check that \( \vec{E}' \) and \( \vec{B}' \) obtained from \( \vec{A}' \)s and \( V' \) are equal to \( \vec{E} \) and \( \vec{B} \). The invariance of the field under such gauge transformations is called gauge invariance. This gauge freedom will be exploited below to obtain simpler equations for \( V \) and \( \vec{A} \).

Equations obeyed by \( V \) and \( \vec{A} \): Substituting equations (1.1) and (1.2) into

\[
\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{1.5}
\]

and confining our attention to the vacuum we have

\[
\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \right).
\]

\(^1\)Recall our discussion of the Aharonov-Bohm effect.
Using vector identities and rearranging terms we obtain
\[ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} + \vec{\nabla} \left( \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) . \]

The left-hand side looks like part of a wave equation and the right-hand side contains the source and an extra, unpleasant term. We use the gauge freedom defined earlier to set the term equal to zero. This particular choice of gauge which makes
\[ \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \] (1.6)
is known as the Lorentz gauge.\(^3\)

Thus, we have in the Lorentz gauge
\[
\begin{align*}
\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{j} \\
\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} &= -\frac{1}{\epsilon_0} \rho .
\end{align*}
\]

Please verify the second of the above equations explicitly starting from Gauss’ law.

**Solution of the wave equation with sources**: We must now solve these wave equations given the spatial and temporal dependence of the charge and current sources. We recall that the solution of Poisson’s equation
\[ \nabla^2 V(\vec{r}) = -\frac{1}{\epsilon_0} \rho(\vec{r}) \]
is given by
\[ V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} . \] (1.7)

\(^2\)The operator
\[ \Box^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \]
is known as the d’Alembertian after Jean Le Rond d’Alembert(1717-1783).

\(^3\)Suppose we have a set of potentials; the question is whether we can choose \(\chi(\vec{r}, t)\) such that the term vanishes. This means that
\[ \vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2} \frac{\partial V'}{\partial t} = 0 . \]
So substituting the gauge transformations in we have
\[ \nabla^2 \chi - \frac{1}{c^2} \frac{\partial \chi}{\partial t} = -\vec{\nabla} \cdot \vec{A} - \frac{1}{c^2} \frac{\partial V}{\partial t} . \]

In the original gauge the right-hand side is non-zero. We know what it is. So we just have to solve the scalar wave equation with this as the source and theorems guarantee the existence of \(\chi\).
Please explain this solution in words based on Coulomb’s Law. Recall, that mathematically the solution can be verified by applying the operator $\nabla^2_r$ to both sides and remembering the result,

$$\nabla^2_r \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta^3(\vec{r} - \vec{r}').$$

The solution to the wave equation with the source is given by$^4$

$$V(\vec{r}, t) = \frac{1}{4\pi \varepsilon_0} \int d^3r' \frac{\rho(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|}. \quad (1.8)$$

In words, the potential at the observation point $\vec{r}$ at time $t$ is given by the Coulomb formula for the charge at an earlier time (the delay is precisely the time it take light to travel from the source to the observer!) integrated over all the sources. This potential is called the \textit{retarded} potential. This is an amazing result and provides hints of the intimate connection between special relativity and electrodynamics. Lest you think this is obvious this holds only for the potentials and not the fields. For example, in electrostatics we have

$$\vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int d^3r' \frac{\vec{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|^3}. \quad \text{not given by the “retarded” version of this as the potentials are!}$$

The electric field for time-dependent sources is \textbf{not} given by the “retarded” version of this as the potentials are!

The vector potentials are given similarly by

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|}. \quad (1.9)$$

We have to learn to evaluate the integral. First we will focus on currents with a single frequency. Since the equations are linear we can use Fourier analysis and superpose the potentials (and fields) due to different frequencies, i.e, integrate over $\omega$. So we write

$$\vec{J}(\vec{r}, t) = \vec{J}(\vec{r}) e^{-i\omega_0 t}, \quad (1.10)$$

note that we use the same symbol for the current which depends on time and the current which depends on space only. Substituting this into equation (1.9) and defining $\omega_0/c = k$ we obtain (be sure about this)

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} e^{-i\omega_0 t} \int d^3r' \frac{\vec{J}(\vec{r}') e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|}. \quad (1.11)$$

$^4$I will provide a verification later or you can read the book
A key point to note is that a current source with a frequency $\omega_0$ leads to a vector potential (and thence, the fields) with the same (sinusoidal) harmonic frequency dependence.

In order to simplify the spatial integral we note that there are three length scales in it: (1) $r \equiv |\vec{r}|$ the distance of the observer from the source, (2) $d = \max \{ |\vec{r}'| \}$ the size of the source and (3) $\lambda = 2\pi/k$ the wavelength of the electromagnetic field.

We will first focus on the far or radiation zone where $r \gg d, \lambda$ without specifying the relative sizes of $d$ and $\lambda$. We then expand the factors with $|\vec{r} - \vec{r}'|$, we have to keep more terms in the exponential than in the denominator. Why? We use the notation $\vec{r} = r \hat{n}$ and write

$$
|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2r\vec{r} \cdot \hat{n}} = r \left(1 - \frac{\vec{r}' \cdot \hat{n}}{r} + \frac{r'^2}{r^2}\right)^{1/2}
$$

to second order in $r'^2/r^2$. We approximate $\exp(ik|\vec{r} - \vec{r}'|)$ by

$$
\exp(ik|\vec{r} - \vec{r}'|) \approx \exp(-ik\vec{r}' \cdot \hat{n})
$$

which implies that we are neglecting terms of the order $kd^2/r$ where we have replaced $r'$ by the maximum value of $r'$, i.e., $d$. Thus we have assumed $r \gg d^2/\lambda$. The other factor $1/|\vec{r} - \vec{r}'|$ can be replaced by $1/r$ since we are interested in fields which decay as $1/r$ which is the definition of the radiation field.\(^5\) Note that in the oscillating factor we have to retain one extra term. Substituting these expansions in the expression for $\vec{A}$ we obtain in the radiation zone

$$
\vec{A}(\vec{r}, t) = \mu_0 \frac{\epsilon_{kr - i\omega_0 t}}{4\pi} \int d^3\vec{r}' \vec{J}(\vec{r}') \exp(ik\vec{r}' \cdot \hat{n}).
$$

This is the key formula. Note that all the angular dependence and the dependence on the details of the current source is in the integral. If we use spherical polar coordinates we have

$$
\hat{n} = \{\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\}
$$

\(^5\)Just to recapitulate we have

$$
|\vec{r} - \vec{r}'|^{-1/2} = (r^2 + r'^2 - 2r\vec{r}' \cdot \hat{n})^{-1/2} = \frac{1}{r} \left(1 - \frac{\vec{r}' \cdot \hat{n}}{r} + \frac{r'^2}{r^2}\right)^{-1/2}
$$

$$
\approx \frac{1}{r} \left(1 + \frac{\vec{r}' \cdot \hat{n}}{r} - \frac{1}{2} \left(\frac{r'}{r}\right)^2 + 3 \left(\frac{\vec{r}' \cdot \hat{n}}{r}\right)^2\right)
$$

(1.14) to second order in $r'^2/r^2$. 

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and we can write
\[ \tilde{f}(\theta, \phi) \equiv \int d^3r' \tilde{J}(\vec{r}') e^{ik\vec{r}'\cdot\hat{n}}. \] (1.16)

Thus we can write
\[ \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{ikr - i\omega_0 t}}{r} \tilde{f}(\theta, \phi). \] (1.17)

So we have reduced the complete integral in Equation (1.11) to a simpler integral in \( \tilde{f} \) in the radiation zone. Observe that we simply have to evaluate \( \tilde{f}(\theta, \phi) \) for each current source distribution and all the quantities can be calculated. We will work out several examples.

**Electric and magnetic field:** Before working out the examples we will determine the fields given the vector potential. The key point to remember is that only terms of order \( 1/r \) need be retained. Since the vector potential is a product of a scalar function times a vector function we use the result
\[ \vec{\nabla} \times \left( g(r) \vec{f}(\vec{r}) \right) = \vec{\nabla} g \times \vec{f} + g \vec{\nabla} \times \vec{f}. \]

In our problem
\[ g = \frac{\mu_0}{4\pi} \frac{e^{ikr - i\omega_0 t}}{r}. \]

First we note that the curl operator in spherical coordinates contains a factor of \( 1/r \) and this along with the \( 1/r \) term in \( g \) yields higher-order terms which can be neglected. So banish \( \vec{\nabla} \times \vec{f} \). Now for \( \vec{\nabla}g \). Since in our problem \( g \) is a function of \( r \) only (no angular dependence) and \( \vec{\nabla} = \hat{n} \partial/\partial r \) we obtain
\[ \vec{\nabla}g = \frac{\mu_0}{4\pi} \hat{n} \frac{\partial}{\partial r} \frac{e^{ikr - i\omega_0 t}}{r} \approx ik \frac{\mu_0}{4\pi} \hat{n} \frac{e^{ikr - i\omega_0 t}}{r} \]
where we have only differentiated the exponential term and not the \( 1/r \) term. Why?

So finally, we have
\[ \vec{B} = ik \frac{\mu_0}{4\pi} \frac{e^{ikr - i\omega_0 t}}{r} \hat{n} \times \vec{f}. \] (1.18)

The magnetic field is perpendicular to the direction of propagation (toward the observation point) \( \hat{n} \).

We can evaluate \( \vec{E} \) using the potentials but then we have to solve for \( V \). Instead we will use
\[ \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \]
which holds in the radiation zone (there are no sources since \( d \ll r \)) and write for the sinusoidal time dependence we have
\[ \vec{E} = \frac{ic}{k} \vec{\nabla} \times \vec{B}. \]
So again we have the same problem of taking the curl of a scalar function times a vector function that we solved when we took the curl of $\vec{A}$. This will be assigned as a homework problem for next week. We find

$$\vec{E} = c \vec{B} \times \hat{n}. \quad (1.19)$$

Thus the electric field is orthogonal to $\vec{B}$ and $\hat{n}$ as we saw for plane waves. Thus harmonically (in time) varying current sources yield harmonic, spherical, electromagnetic waves!

So the problem is reduced to finding $\vec{f}(\theta, \phi)$ given $\vec{J}(\vec{r})$ (the harmonic time dependence has been factored out.) We will study some examples next as promised.