FOURIER ANALYSIS: an informal summary

Consider a function $f(x)$ that is nonzero in an interval $[-a, a]$. We are only interested in functions defined in this interval. It can be represented as

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos\left(\frac{2\pi}{2a} nx\right) + B_n \sin\left(\frac{2\pi}{2a} nx\right)].$$

(1.1)

The numbers $\{A_n, B_n\}$ are called Fourier coefficients. The factor $2a$ has been placed so as to emphasize that it is the length of the interval. Note that we have made $f(x)$ automatically a periodic function with period $2a$.

Given the function $f(x)$ its Fourier coefficients can be determined because the trigonometric functions form an orthogonal basis. In particular, $\cos(\frac{2\pi}{2a} mx)$ is orthogonal to $\cos(\frac{2\pi}{2a} nx)$ for $m \neq n$ and also to $\sin(\frac{2\pi}{2a} mx)$ for any $m$. What this means is that the appropriate inner product vanishes:

$$I_{mn} \equiv \int_{-a}^{a} dx \cos\left(\frac{2\pi}{2a} nx\right) \cos\left(\frac{2\pi}{2a} mx\right)$$

(1.2)

vanishes for $m \neq n$. This can be verified using elementary trigonometric identities. First change variables from $x$ to $y = (2\pi/2a)x$ obtaining

$$I_{mn} \equiv \left(\frac{a}{\pi}\right) \int_{-\pi}^{\pi} dy \cos(ny) \cos(my).$$

(1.3)

Note that

$$\cos(ny) \cos(my) = \frac{1}{2} [\cos((m + n)y) + \cos((m - n)y)].$$

(1.4)

Substituting this into the integral it is easy to see that it vanishes for $m \neq n$ because the integral of the cosine over the interval $-\pi$ to $+\pi$ vanishes. For $m = n$, we find

$$I_{nn} = \int_{-a}^{a} dx \cos^2(n\pi x/a) = a.$$

We write this as $I_{nn} = a\delta_{nn}$. You should check that

$$J_{mn} \equiv \int_{-a}^{a} dx \cos\left(\frac{2\pi}{2a} nx\right) \sin\left(\frac{2\pi}{2a} mx\right)$$

(1.5)

vanishes. You will need

$$\cos(ny) \sin(my) = \frac{1}{2} [\sin((m + n)y) - \sin((m - n)y)].$$

(1.6)

We can also show similarly that

$$I_{mn} = \int_{-a}^{a} dx \sin\left(\frac{2\pi}{2a} nx\right) \sin\left(\frac{2\pi}{2a} mx\right) = a\delta_{mn}.$$

(1.7)
Given \( f(x) \) how does one obtain the coefficients \( A_n \) and \( B_n \)? We simply project out the component along the appropriate direction as usual. This is exactly as if we were given
\[
\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}
\]
and we could find \( v_2 \) by taking the dot product of this equation with \( \hat{j} \) and find \( v_2 = \vec{v} \cdot \hat{j} \).

We now multiply both sides of Equation 1.1 by \( \cos \left( \frac{2\pi}{2a} mx \right) \) for \( m \geq 1 \) and integrate from \(-a\) to \( a\) and obtain
\[
A_m = \frac{1}{a} \int_{-a}^{a} dx f(x) \cos \left( \frac{2\pi}{2a} mx \right).
\] (1.8)

Similarly, by multiplying both sides of Equation 1.1 by \( \sin \left( \frac{2\pi}{2a} mx \right) \) and integrating from \(-a\) to \( a\), we obtain,
\[
B_m = \frac{1}{a} \int_{-a}^{a} dx f(x) \sin \left( \frac{2\pi}{2a} mx \right).
\] (1.9)

Please find an expression for \( A_0 \).

Therefore, in summary,

\[
f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{\pi n x}{a} \right) + B_n \sin \left( \frac{\pi n x}{a} \right) \right],
\]

where the Fourier coefficients are given by
\[
A_m = \frac{1}{a} \int_{-a}^{a} dx f(x) \cos \left( \frac{\pi n x}{a} \right),
\]
and
\[
B_m = \frac{1}{a} \int_{-a}^{a} dx f(x) \sin \left( \frac{\pi n x}{a} \right).
\]

One way to think about this expansion: A three-dimensional vector can be expanded in terms of its three components along the \( x \), \( y \), and \( z \) directions. Here we are doing the astonishing generalization of expanding all (nice) functions! We find that we need a lot more than three components (not terribly surprising), in fact an infinite number. The orthogonal basis corresponding to \( \hat{i}, \hat{j}, \hat{k} \) are the sine and cosine functions with different wavelengths.

The figures show examples of how well the Fourier series approximate different functions. I have used the interval \([0, 2\pi]\).

The function \( f_1(x) \) which is a continuous, piecewise linear function defined by
\[
f_1(x) = \begin{cases} 
  x & \text{for } 0 \leq x \leq \pi \\
  2\pi - x & \text{for } \pi \leq x \leq 2\pi
\end{cases}
\]
(1.10)
can be expanded in a Fourier series. The details of the integrals required to evaluate
the coefficients are skipped. You can use Mathematica clumsily to do this and the file
is attached. We find

\[ f_1(x) = \pi/2 - 4 \sum_{\text{odd } m} \frac{1}{m^2} \cos mx. \]  \hspace{1cm} (1.12)

In Figure 1 you will find plots of the sum of the constant plus the first three terms and
also the constant plus the first forty terms. Note how fast the series converges for this
function with a discontinuity in the slope.

We consider next the function \( f_2(x) \) which is a piecewise constant function, defined by

\[ f_2(x) = \begin{cases} 
1 & \text{for } 0 < x < \pi \\
-1 & \text{for } \pi < x < 2\pi.
\end{cases} \]  \hspace{1cm} (1.13)

This can be expanded in a Fourier series as

\[ f_2(x) = 4 \sum_{\text{odd } m} \frac{1}{m} \sin mx. \]  \hspace{1cm} (1.15)

In Figure 2 you will find plots of the first three terms in the sum, the sum of the
first 100 terms and also the first 400 terms. Again note that even a function with a
discontinuity is rather well approximated with 100 terms except at the position of the
discontinuity; there is an overshoot and an undershoot on either side. This is referred
to as the Gibbs phenomenon; remarks in lecture.

Sometimes is more convenient to use exponentials and write

\[ f(x) = \sum_{n=-\infty}^{\infty} f_n e^{in\pi x/a}. \]  \hspace{1cm} (1.16)

Note the range of \( n \) here in contrast to that in the expansion in terms of sines and
cosines. We can relate \( f_n \) to \( A_n \) and \( B_n \). Note that the set \{exp(i\pi mx/a)\} forms an
orthogonal set. What does this mean? For different values of \( m \) and \( n \), taking the
complex conjugate of one and multiplying by the other and integrating from \(-a\) to \(a\)
yields 0 for \( m \neq n \):

\[ \int_{-a}^{a} dx e^{-im\pi x/a} e^{in\pi x/a} = \int_{-a}^{a} dx e^{i(n-m)\pi x/a} = \frac{a}{i\pi(n-m)} \left( e^{i(n-m)} - e^{-i(n-m)} \right) \]

\[ \propto \left( e^{2i\pi(n-m)} - 1 \right) = 0 \]  \hspace{1cm} (1.17)

for \( m \neq n \) since \( \exp(i2N\pi) \) is 1 for any integer \( N \). Clearly, if \( m = n \) we have

\[ \int_{-a}^{a} dx e^{-in\pi x/a} e^{in\pi x/a} = \int_{-a}^{a} dx = 2a. \]  \hspace{1cm} (1.18)

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Thus we write
\[ \int_{-a}^{a} dx e^{-i\pi nx/a} e^{i\pi mx/a} = 2a \delta_{mn} \] (1.19)
where \( \delta_{mn} \) is the usual Kronecker delta function. This means as can be seen in a rather straightforward fashion that
\[ f_n = \frac{1}{2a} \int_{-a}^{a} dx e^{-i\pi nx/a} f(x) ; \] (1.20)

As an aside we note that
\[ f(x) = f_0 + \sum_{n=1}^{\infty} (f_n + f_{-n}) \cos \left( \frac{n\pi x}{a} \right) + \sum_{n=1}^{\infty} i(f_n - f_{-n}) \sin \left( \frac{n\pi x}{a} \right) \]
which implies
\[ A_n = f_n + f_{-n} \text{ and } B_n = i(f_n - f_{-n}) . \]

Note that since \( f(x) \) is real we have \( f^*(x) = f(x) \). Taking the complex conjugate of the expression in Equation(1.16) we obtain
\[ f^*(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{-i\pi nx/a} . \]

Changing \( n \) to \( -n \) (dummy variable in the sum) we have
\[ f^*(x) = \sum_{n} f_{-n}^* e^{i\pi nx/a} \]
which should equal \( f(x) \) and comparing coefficients we obtain the useful result \( f_{-n} = f_n^* \).
Now we study the limit $a \to \infty$ heuristically. We will change from the discrete variable $n$ to a continuous variable $k$ (the wavevector with units/dimensions $L^{-1}$) defined by

$$k \equiv n \frac{\pi}{a} .$$

This implies

$$\Delta k = \Delta n \frac{\pi}{a} = \frac{\pi}{a}$$

which becomes infinitesimal as $a \to \infty$. We will assume that the integral

$$\int_{-a}^{a} dx e^{-i\pi nx/a} f(x) \to \int_{-\infty}^{\infty} dx e^{-ikx} f(x)$$

exists. Therefore, we define

$$F(k) = \lim_{a \to \infty} 2a f_n$$

and we have

$$F(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx} .$$

(1.21)

Now we rewrite the sum on $n$ into an integral over the continuous variable $k$:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{in\pi x/a} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\pi}{a} [2a f_n] e^{ikx}$$

(1.22)

where we have put in the factor of $2a$ with $f_n$ so that it goes over to $F(k)$ when the $a \to \infty$ limit is taken and the factor $\pi/a$ is $\Delta k$ which goes over to $dk$ and we have

$$\to \frac{1}{2\pi} \int_{-\infty}^{\infty} dk F(k) e^{ikx} .$$

(1.23)

Thus we obtain

\[
\begin{align*}
  f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} F(k) e^{ikx} \\
  \text{with the inverse Fourier transform given by} \\
  F(k) &= \int_{-\infty}^{\infty} dx f(x) e^{-ikx}
\end{align*}
\]

We next introduce the concept of a *delta function* also known as the Dirac delta function. We substitute one of the above equations into the other remembering that the integration variable is a dummy variable:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dx' f(x') e^{-ikx'}$$

(1.24)
We interchange the orders of integration (‘aye, therein lies the rub” as the mathematician would say to my deaf ears) and obtain

\[ f(x) = \int_{-\infty}^{\infty} dx' f(x') \left[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \right] \]

Let us call the expression within square brackets which depends on \(x - x', \delta(x - x');\) we have

\[ f(x) = \int_{-\infty}^{\infty} dx' f(x') \delta(x - x') \]

So clearly what \(\delta(x - x')\) does in the integral over \(x'\) is to pick out the value of \(f\) at \(x\).

The above equation can be viewed as the definition of the delta function along with

\[ \delta(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \quad (1.25) \]