We wish to consider two protons which are stationary in the lab frame $S$ separated by a distance $a$ along the $y$-axis. In this frame the force is given by Coulomb's law $\hat{y} e^2 / (4\pi\epsilon_0 a^2)$. What is the force in a frame $S'$ moving with a velocity $v$ with respect to $S$? We wish to find this force (a) by learning how forces transform under Lorentz boosts and (b) explicitly calculating the forces from the fields in $S'$ from our recently acquired knowledge of the transformation of fields.

First how do forces transform? We will restrict ourselves to the case in which the particle (on which the force is acting) in instantaneously at rest and find the force in moving frame.$^1$ Let us find this heuristically, by considering a particle of (rest) mass $m$ at rest in the frame $S$. At time $t = 0$ a force $F_x$ acts along the $x$-axis for a time $\Delta t$. The momentum (along the $x$-axis) at the end of that period is $F_x \Delta t$. We will use the definition of the force as the rate of change of momentum to obtain its transformation properties. So let us write down the transformation equations in the following form:

$$x' = \gamma [x - \beta (ct)], \quad y' = y, \quad z' = z, \quad \text{and} \quad ct' = \gamma [ct - \beta x]. \quad (1.1)$$

Since $(\vec{p}, E/c)$ form a four-vector we can (automatically) write the corresponding transformations

$$p'_x = \gamma [p_x - \beta E/c], \quad p'_y = p_y, \quad p'_z = p_z \quad \text{and} \quad E' = \gamma [E - \beta cp_x]. \quad (1.2)$$

The force in the primed frame is the change in the momentum as measured in that frame divided by the time interval measured by clocks in the same frame:

$$F'_x = \frac{\Delta p'_x}{\Delta t'} \quad (1.3)$$

in the limit $\Delta t' \to 0$. From the transformations we have

$$\Delta t' = \gamma [\Delta t - \beta/ c \Delta x] \quad \text{and} \quad \Delta p'_x = \gamma [\Delta p_x - \beta/ c \Delta E];$$

We note that since the particle started from rest the distance moved is

$$\Delta x = \frac{1}{2} at^2 = \frac{1}{2} \frac{F_x}{2 m} (\Delta t)^2$$

which is higher-order in $\Delta t$. Similarly, the change in energy $\Delta E$ (we will assume that the particle acquires non-relativistic velocities; the transformation velocity can be large) is $1/2 (\Delta p)^2 / m$ which is also of order $(\Delta t)^2$. Thus we have to order $\Delta t$,

$$\Delta t' \approx \gamma \Delta t \quad \text{and} \quad \Delta p'_x \approx \gamma \Delta p_x .$$

$^1$The general case is considered in Griffiths 12.2.4; note that he points out that only in the case we consider are the equations “reasonably tractable.”
Therefore,

\[ F'_x = \frac{\Delta p'_x}{\Delta t'} = \frac{\Delta p_x}{\Delta t} = F_x. \]  

(1.4)

We obtain, therefore, that the force along the direction of the frame motion is the same in the moving frame as it is in the rest frame.

For the case of \( F'_y \) we note that \( \Delta p'_y = \Delta p_y \) since the perpendicular directions are unaffected. Thus we obtain

\[ F'_y = \frac{F_y}{\gamma}. \]

Thus, the force in directions perpendicular to the velocity of the frame is smaller by a factor of \( \gamma \) compared to that in the rest frame.

We now recall that the field component parallel to the velocity is unaltered while the perpendicular components are enhanced by the factor \( \gamma \).

This presents an apparent paradox for the problem of two protons. In the rest frame we have the Coulomb field along \( \hat{y} \) and the corresponding force. In the moving frame the force along \( \hat{y} \) (perpendicular to the Lorentz boost) is reduced by a factor \( 1/\gamma \) while the electric field along \( \hat{y} \) is enhanced by a factor of \( \gamma \).

The resolution of this depends on the fact that in the frame \( S' \) the proton at \( y = 0 \) is moving and hence, gives rise to a magnetic field which affects the proton at \( y = a \). We can simply use the results we derived for the fields of a charge moving with a velocity \( v\hat{x} \): The magnetic field \( \vec{B}' \) in \( S' \) is given by \( \frac{1}{c^2} \vec{v} \times \vec{E}' \) where \( \vec{E}' \) is the electric field in \( S' \). Given that \( \vec{v} = v\hat{x} \) and \( \vec{E}' = \hat{y} \gamma E \) where \( E = e/(4\pi \epsilon_0 a^2) \) is the magnitude of the field in \( S \) we have

\[ \vec{B}' = \hat{z} \gamma \frac{v}{c^2} E. \]

So the force on the proton at \( y = a \) due to this field is given by the Lorentz force Law:

\[ \vec{F}' = e \left[ \vec{E}' + v\hat{x} \times \vec{B}' \right] = e \left( \hat{y} \gamma E - \hat{y} \frac{v^2}{c^2} \gamma E \right) \]

\[ = \hat{y} eE \left( \gamma (1 - \frac{v^2}{c^2}) \right) = \hat{y} \frac{eE}{\gamma} = \hat{y} \frac{F}{\gamma}. \]

(1.5)

This is precisely the transformation law for the forces! So even though the electric field along \( \hat{y} \) is enhanced by a factor of \( \gamma \) (with respect to the field in \( S \)) in \( S' \), the effect of the magnetic field in \( S' \) and the associated Lorentz force reduces the net force so that the force in \( S' \) is in fact reduced by a factor of \( 1/\gamma \) as expected form general considerations about forces.