Transverse magnetic waves in a rectangular wave guide

Propagation of electromagnetic waves confined inside metallic boundaries is a subject of considerable practical importance. Typically, the wavelengths are comparable to the dimensions of the wave guides. In practice, the fields penetrate into the conducting walls (they are attenuated exponentially with a characteristic length called the skin depth) and there are consequent (ohmic) dissipative losses. The complete problem is difficult to solve and we will focus on obtaining the dispersion relation. Consider a wave guide with a rectangular cross section (oriented along the cartesian axes); the axis of the wave guide is along the \( z \)-direction which is the direction of propagation of the wave.

Recall that the tangential components of the electric field and the normal component of the magnetic field are continuous. We will assume a perfect conductor so that the tangential component of \( E \) and the normal component of \( B \) vanish inside the conductor and hence, also just outside the conductor by continuity.

We look for solutions that vary as \( e^{ikz-i\omega t} \) and focus on transverse magnetic modes, i.e., \( H_z = 0 \); only components of \( H \) transverse to the direction of propagation \( z \) are non-vanishing. We will assume that the medium inside the conducting walls is linear and can be characterized by \( \epsilon \) and \( \mu \); you can assume that the waves propagate in a vacuum if you wish.

We will use the compact abbreviations \( \partial_x = \partial/\partial x \), etc. From the generalization of Ampere’s Law

\[
\vec{\nabla} \times \vec{H} = \partial_t \vec{D} = \epsilon \partial_t \vec{E}
\]

the \( x \)-component yields using \( \partial_t = -i\omega \),

\[
\partial_y H_z - \partial_z H_y = -i\omega \epsilon E_x.
\]

This yields upon using \( H_z = 0 \) since we have a transverse magnetic wave and replacing \( \partial_z \) by \( ik \)

\[
H_y = \frac{\omega \epsilon}{k} E_x \tag{1.1}
\]

The \( y \)-component of the same Maxwell’s equation

\[
\partial_z H_x - \partial_x H_z = -i\omega \epsilon E_y
\]

yields using the same ideas (\( H_z = 0 \) and \( \partial_z = ik \))

\[
H_x = -\frac{\omega \epsilon}{k} E_y \tag{1.2}
\]

From Faraday’s Law

\[
\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t = -\mu \partial \vec{H}/\partial t,
\]

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we obtain for the $x$-component

$$\partial_y E_z - \partial_z E_y = -\mu \partial_t H_x .$$

Now we use $\partial_t = -i\omega$ and substitute the expression for $H_x$ from Equation(1.2) and find

$$\partial_y E_z - iKE_y = i\mu\omega \left( -\frac{\omega\epsilon}{k} E_y \right).$$

This yields upon multiplying by $ik$ and rearranging

$$E_y = \frac{ik\partial_y E_z}{\mu\epsilon\omega^2 - k^2} . \quad (1.3)$$

Similarly, from the $y$-component we obtain from

$$\partial_z E_x - \partial_x E_z = -\partial_t B_y = -\mu \partial_t H_y$$

the relation

$$ikE_x - \partial_x E_z = i\omega\mu H_y$$

and upon substituting for $H_y$ from Equation(1.1)

$$ikE_x - \partial_x E_z = i\frac{\omega^2\epsilon\mu}{k} E_x .$$

$$E_x = \frac{ik\partial_x E_z}{\mu\epsilon\omega^2 - k^2} . \quad (1.4)$$

Thus, we have shown that $E_x$, $E_y$, $H_x$, and $H_y$ can be obtained from $E_z$ which satisfies the wave equation with appropriate boundary conditions:

$$(\partial^2_x + \partial^2_y + \partial^2_z) E_z = \mu\epsilon \partial^2_t E_z \quad (1.5)$$

which can be written upon substituting the ansatz

$$E_z(x, y, z, t) = E_0(x, y) e^{ikz-i\omega t}$$

and cancelling the exponential factor as

$$\partial^2_x E_0 + \partial^2_y E_0 - k^2 E_0 = -\mu\epsilon\omega^2 E_0 .$$

So we have to solve the equation obeyed by $E_0$:

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] E_0(x, y) = -\gamma^2 E_0(x, y) \quad (1.6)$$

where $\gamma^2 \equiv \mu\epsilon\omega^2 - k^2$. Since the tangential component of the electric field has to vanish at the boundaries $E_z$ must vanish at $x = 0, a$ and $y = 0, b$ at all times and therefore, we have

$$E_0(x, y) = E_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) . \quad (1.7)$$
The differential equation yields
\[-\frac{m^2\pi^2a^2}{a^2} - \frac{n^2\pi^2}{b^2} = -\gamma^2 = -\left(\mu\epsilon\omega^2 - k^2\right).\]

This gives rise to the dispersion relation
\[\epsilon\mu\omega^2 = k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}.\]  

(1.8)

Because of the sines instead of the cosines the lowest TM mode is (11). We assemble all the fields together apart from a factor of \(e^{ikz-i\omega t}\):

\[H_z = 0\]  

(1.9)

\[E_z = E_{11} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)\]  

(1.10)

\[E_y = E_{11} \frac{ik}{\gamma^2} \frac{\pi}{b} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right)\]  

(1.11)

\[E_x = E_{11} \frac{ik}{\gamma^2} \frac{\pi}{a} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)\]  

(1.12)

\[H_x = -E_{11} \frac{i\omega\epsilon}{\gamma^2} \frac{\pi}{b} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right)\]  

(1.13)

\[H_y = E_{11} \frac{i\omega\epsilon}{\gamma^2} \frac{\pi}{a} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)\]  

(1.14)

where the factors of \(i\) when they appear are interpreted as usual. The real part yields \(\cos(kz-\omega t)\) if there is no \(i\) and \(\mp\sin(kz - \omega t)\) if one has \(\pm i\).

Please check for signs or use them in the HW and blame them on me!