You are allowed to use Mathematica or equivalent to do integrals.

2.1) Compute
\[ A^2 \int_{-a}^{a} dx (a^2 - x^2)^2 = A^2 \frac{16a^5}{15}. \]
Therefore, setting this to unity \[ A = \frac{\sqrt{15}}{4a^{5/2}}. \]

\[ \langle x \rangle = 0 \] since the integrand is an odd function.

\[ \langle p \rangle = 0 \] since the wave function is real. One can also argue that the integrand is odd since \( \psi \) is even, its derivative is odd.

\[ \langle x^2 \rangle = \int_{-a}^{a} dx \frac{\sqrt{15}}{4a^{5/2}} (a^2 - x^2) \frac{x^2}{4a^{5/2}} (a^2 - x^2) = \frac{15}{16a^5} \int_{-a}^{a} dx (a^2 - x^2)^2 x^2 = \frac{a^2}{2}. \]

\[ \langle p^2 \rangle = \hbar^2 \int_{-a}^{a} \left[ \frac{d}{dx} A(a^2 - x^2) \right] \left[ \frac{d}{dx} A(a^2 - x^2) \right] dx = \hbar^2 \frac{15}{16a^5} \int_{-a}^{a} dx 4x^2 = \frac{5\hbar^2}{2a^2}. \]

Clearly \( \sigma_x = \frac{a}{\sqrt{7}} \) and \( \sigma_p = \sqrt{\frac{5}{2}} \frac{\hbar}{a}. \)

Thus we obtain
\[ \sigma_x \times \sigma_p = \sqrt{\frac{5}{14}} \hbar = \sqrt{\frac{10}{7}} \frac{\hbar}{2} > \frac{\hbar}{2}. \]
2.2) We substitute the given form of Ψ, Ψ(ℏ, ℏ) = A e^{ikx} e^{-iEt/ℏ} in the Schrödinger equation for a free particle (set V(ℏ) = 0.)

\[ i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_x^2 \Psi. \]

We will often use \( \partial_t \) for \( \frac{\partial}{\partial t} \) etc.

Evaluate the required derivatives:

\[ \partial_t \Psi = -\frac{iE}{\hbar} \Psi \Rightarrow i\hbar \partial_t \Psi = E\Psi. \]

(Pay attention to the signs when you are dealing with \( i \)'s; we have \( i \times (-i) = 1 \) above.

\[ \partial_x \Psi = ik\Psi \text{ and } \partial_x^2 \Psi = (ik)^2 \Psi \Rightarrow -\frac{\hbar^2}{2m} \partial_x^2 \Psi = \frac{\hbar^2 k^2}{2m} \Psi. \]

Substituting into the Schrödinger equation for the free particle we see that the given wave function satisfies this equation with the dispersion relation \( E = \hbar^2 k^2/(2m) \).

We have \( \Psi \) given by

\[ A \exp \left[i \left(-\frac{Et}{\hbar} + kx\right)\right] \]

and therefore \( |\Psi|^2 = A^2 \). Please remember that the modulus of \( e^{if(x,t)} \) is one for real \( f \).

If we try to normalize \( \Psi \) we find the integral \( \int_{-\infty}^{\infty} dx \) diverges. It is embarrassing that we run into trouble with the simplest problem. There are two ways around this that will be discussed later in the course.

2.3) Use Mathematica to evaluate the integral

\[ A_0^2 \int_{-\infty}^{\infty} dx \ e^{-2\lambda x^2} = A_0^2 \times \sqrt{\frac{\pi}{2\lambda}}. \]

Therefore, \( A_0 = (2\lambda/\pi)^{1/4} \).
\( \langle x \rangle \) vanishes since the \(|\Psi(x, t)|^2\) is an even function of \(x\) and \(\langle p \rangle\) vanishes since the spatial part of the wave function is real. For \(\langle x^2 \rangle\)

\[
\int_{-\infty}^{\infty} dx \ x^2 A_0^2 e^{-2\lambda x^2} = \frac{1}{4\lambda}.
\]

You should have guessed the \(\lambda\) dependence.

Compute \(\langle p^2 \rangle\) by using the expression where you have performed integration by parts once:

\[
\langle p^2 \rangle = h^2 \int_{-\infty}^{\infty} dx \ \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} = A_0^2 h^2 \int_{-\infty}^{\infty} dx \left( \frac{d e^{-\lambda x^2}}{d x} \right)^2 = \lambda h^2.
\]

You can use the following to do the integral on Mathematica:

\[
\text{Integrate}\[\sqrt{2 \text{lambda}/\text{Pi}} \ hbar^2 \ (D[\text{Exp}[-\text{lambda} \ x^2], x])^2, \{x, -\infty, \infty\}, \text{Assumptions} \rightarrow \{\text{Im}[\text{lambda}] == 0, \text{Re}[\text{lambda}] > 0\}]
\]

We have \(\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{4\lambda}\) and \(\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = h^2 \lambda\) and therefore, \(\sigma_x \sigma_p = \hbar/2\). The uncertainty relation is satisfied as an equality for Gaussian wave functions, a useful fact. Finding the most general wave function that satisfies the uncertainty relation as an equality is a much harder problem.

2.4) The dimensions of \(A\) are given by those of \(\Psi\): \(L^{-1/2}\).
Since \(\lambda \times \) a length is dimensionless \([\lambda] = L^{-1}\).

We have to normalize the wave function; we use the fact that the integrand is even to set

\[
2A^2 \int_{0}^{\infty} dx \ e^{-2\lambda x} = 1
\]

to obtain \(A = \sqrt{\lambda}\). The probability of finding a particle between \(-\lambda^{-1}\) and \(\lambda^{-1}\) is the integral of the normalized probability density between the given points:

\[
\int_{-\lambda^{-1}}^{\lambda^{-1}} dx \ A^2 e^{-2\lambda|x|}.
\]

The integral can be done to find \(1 - e^{-2} \approx 0.865\).
There are trivial observations such as time dependence etc. The key difference is that the spatial part of the wave function is not differentiable at $x = 0$.

*Extra information:* Why is this worth noting? The Schrödinger equation involves the second derivative with respect to $x$ and the fact that this does not exist is worrisome. It turns out that the potential that yields such a wave function is singular; it is technically not a function even though it is called a delta function.