Quantum mechanics is a formalism used to describe ensembles of identically prepared systems. We will ignore the problem of how such ensembles are created. Quantum mechanics describes the ensemble so prepared by a wave function $\Psi(x,t)$ which associates a complex numbers, called the probability amplitudes, with each space-time point. For given values of the position $x$ and and time $t$ the real number $|\Psi(x,t)|^2$ is the probability density of finding the particle $x$ at time $t$. In the standard formulation no further knowledge about the systems of the ensemble beyond that in the wave function is possible.

So long as no measurement of any physical quantity is made on the systems of the ensemble (and they are not otherwise disturbed by any external influence), the wave function evolves smoothly and in a way which is completely determined by the physically relevant quantities such as the particle mass, the potential and is described by Schrödinger’s equation.

In quantum mechanics observables are represented by Hermitian operators: energy by the Hamiltonian $\hat{H}$, momentum along $x$ by $-i\hbar \partial / \partial x$, etc. Measurements are made on individual systems of the ensemble, and for any given measured quantity the possible results of the measurement of an observable are the eigenvalues of the operator corresponding to the observable. The probability of obtaining each of the values is determined by the wave function. If the operator is $\hat{A}$ and its (normalized) eigenfunctions are $\phi_j$ with eigenvalues $a_j$ then we expand the given wave function in terms of the complete orthonormal set $\{\phi_j\}$:

$$\psi = \sum c_j \phi_j \cdotp$$

The quantity $|c_j|^2$ is the probability that you obtain the value $a_j$. This will be illustrated for $2 \times 2$ matrices. As in the case of energy measurement the coefficients are obtained by projecting out the required coefficient. If you want $c_i$ multiply by $\phi_i^*(x)$ and integrate. By using orthonormality we find

$$c_i = \int_{-\infty}^{\infty} dx \phi_i^*(x) \psi(x) \cdotp$$

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1Say what? We will define this soon

2\[ \hat{A} \phi_j = a_j \phi_j. \]
In very special cases the measurement, and the state vector, may be such that one and only one result is seen on all measured systems. In that case the state vector is in an eigenstate of the measured quantity.

Suppose we have drawn our system from an ensemble which at time $t$ is described by a wave function $\Psi$ which is not an eigenstate of some quantity $A$. At time $t$ we make a measurement of $A$ on some of the systems of the ensemble. As noted above, we will get the various possible eigenvalues $a_i$ with some probability $p_i$. What is the correct description immediately after the measurement of the systems which have yielded the result $a_i$? The standard textbook prescription (projection postulate or collapse of the wave function) is that this sub-ensemble is described by a new wave function $\phi_i(x)$ which is the eigenstate of $A$ corresponding to the eigenvalue $a_i$ which was measured i.e. for this set of systems we get a discontinuous jump: The necessity and indeed the meaningfulness of the projection postulate is still a matter of considerable debate.

The collapse postulate is in fact a consequence of an intuitively more satisfactory postulate: that if one makes a measurement twice in rapid succession, one always obtains the same result.
Elementary linear algebra for quantum mechanics:

(This part is the main point of the lecture on the Wednesday before Thanksgiving)

Suppose we have a quantum-mechanical system that can be described by just two complex numbers (not a function but a column vector with two entries.) This means that the system can be in one of two levels only. Such systems exist and are very important: electron spin, proton spin, and many systems can be described by two-level systems.

\[ \psi \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \]

\(|c_1|^2\) is the probability of finding the system in the first level and \(|c_2|^2\) is the probability of finding the system in level 2. This system will evolve in time according to the Schrödinger equation:

\[ i\hbar \frac{d}{dt} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \hat{H} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}. \]  

(2)

In the preceding \(\hat{H}\) is a \(2 \times 2\) matrix. Given the Hamiltonian we have to solve two coupled ordinary differential equations instead of a partial differential equation.

Suppose someone gives you the following putative Hamiltonian:

\[ \hat{H} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}. \]

We decide to find the (energy) eigenvalues determined by

\[ \det |\hat{H} - \lambda I| = 0. \]

We obtain the equation

\[ (1 - \lambda)^2 + 9 = 0 \Rightarrow \lambda = 1 \pm i\sqrt{3}. \]

It has non-vanishing imaginary part. This could not be the result of measuring the energy in a real laboratory. So we have to demand that the Hamiltonian operator (or matrix) has only real eigenvalues.

Next you are given

\[ \hat{H} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \]
We find the (energy) eigenvalues; they are determined by

\[(1 - \lambda)^2 = 0 \Rightarrow \lambda = 1.\]

When two energy eigenvalues are the same they are said to be degenerate\(^3\) Now we find the eigenvectors because we know that in order to find the time evolution of a state we need to expand the wave function (actually vector) in terms of the eigenvectors of \(\hat{H}\). So we look at

\[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix} =
\begin{pmatrix}
a \\
b
\end{pmatrix}.
\]

We have

\[a + 2b = a \quad \text{and} \quad b = b.\]

There is only one eigenvector

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix}.
\]

So we do not get a complete set. Further embarrassment!

If we restrict operators to be Hermitian then the eigenvalues are real and the eigenvectors can be chosen to form a complete orthonormal set. This is the motivation for the restriction to Hermitian operators.

The transpose of a matrix \(A\), denoted by \(A^T\) is defined by

\[(A^T)_{ij} = A_{ji}.\]

Thus we have

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^T
= \begin{pmatrix}
a & c \\
b & d
\end{pmatrix}.
\]

If a matrix is equal to its transpose, i.e., \(A = A^T\) the matrix is said to be symmetric.

For complex matrices (quantum mechanics does involve complex numbers) we can define an operation called the the Hermitian conjugate of a matrix denoted by \(\dagger\): If

\[
A = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \quad \text{then} \quad A^\dagger = \begin{pmatrix}
a^* & c^* \\
b^* & d^*
\end{pmatrix}.
\]

\(^3\)No value judgement is attached to this word.
That is to say we complex conjugate all the elements and transpose the matrix to obtain its Hermitian conjugate.

If \( A = A^\dagger \), \( A \) is said to be Hermitian. Thus

\[
\left(A^\dagger\right)_{ij} = A_{ji}^*.
\]

So \( A_{ii}^\dagger = A_{ii} \), the diagonal elements of a Hermitian matrix are real.

Check that the matrix \( \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \) is Hermitian.

Consider a two-level system described by the Hamiltonian

\[
\hat{H} = \begin{pmatrix} \hbar \omega & 0 \\ 0 & -\hbar \omega \end{pmatrix}.
\]

If the initial state is described by the wave vector \( \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \) what is the wave vector at time \( t \)? same procedure as always. First find the eigenvalues and eigenvectors of \( \hat{H} \); expand the given vector at \( t = 0 \) as a linear combination of the eigenvectors of \( \hat{H} \). Append appropriate phase factors \( e^{-iE_t/\hbar} \).

Since the Hamiltonian is diagonal its eigenvalues are the diagonal elements, to wit, \( \pm \epsilon \).

It is easy to verify that the normalized eigenvectors are

\[
\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Now we need to find \( c_1 \) and \( c_2 \) in

\[
\psi(t = 0) = c_1 \psi_1 + c_2 \psi_2.
\]

\[
\begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

It is easy to find \( c_1 \) and \( c_2 \): \( c_1 = 1/\sqrt{2} \) and \( c_2 = -i/\sqrt{2} \). Thus

\[
\psi(t = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(-\frac{i}{\sqrt{2}}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Now we have from the usual prescription

\[
\psi(t) = e^{-i\omega t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\omega t} \left(-\frac{i}{\sqrt{2}}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Consider an observable represented by the operator
\[ S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \]

The eigenvalues are \( \pm \hbar/2 \) and the corresponding eigenvectors are\(^4\)
\[ \phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \]

Suppose \( \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and we measure \( S_y \). What are the possible values of the results of the measurement and what are the corresponding probabilities. The first part is easy: the possible results of the measurement are the eigenvalues \( \pm \hbar/2 \). To find the probabilities we write
\[ \psi = c_1 \phi_1 + c_2 \phi_2 \]
where we have expanded the given wave function in terms of the complete set of orthonormal eigenfunctions of the operator we are interested in, i.e., \( S_y \). Now \( |c_1|^2 \) is the probability of obtaining \( +\hbar/2 \). How do we find the coefficients? If you substitute the known information we get two equations for \( c_1 \) and \( c_2 \) and we solve them:
\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}. \]

This yields two equations that can be solved for \( c_1 \) and \( c_2 \).

Alternatively, you can project them out. Recall that in the case of one-dimensional problems if \( \{ \psi_n(x) \} \) are the eigenfunctions of the Hamiltonian that form a complete orthonormal set we can write
\[ \psi(x) = \sum_j c_j \psi_j(x). \]
we need \( c_i \) we take the inner product with \( \psi_i \): we multiply by \( \psi_i^*(x) \) and integrate over \( x \):
\[ c_i = \int_{-\infty}^{\infty} dx \psi_i^*(x) \psi(x). \]

**I will change notation and denote the components by arguments in brackets:** the first component of the first eigenvector \( \phi_1 \) is \( \phi_1(1) \), This makes the analogy of

\(^4\text{Please check that these are eigenvectors and that they form an orthonormal set.}\)
the component to \( x \) perhaps more transparent.

We note the following: What is \( \sum_{i=1}^{2} \phi_{1}^{*}(i) \phi_{2}(i) \)? Orthogonal since they are eigenvectors of a Hermitian matrix. What is \( \sum_{i=1}^{2} \phi_{1}^{*}(i) \phi_{1}(i) \)? 1 since it is normalized.

In our problem to find \( c_1 \) we take the inner product of \( \phi_{1} \) with \( \psi \). We take each component of \( \phi_{1}^{*} \) multiply the corresponding component of \( \psi \) and sum over \( i \) (this is the equivalent of integrating over \( x \).)

So we obtain

\[
c_1 = \sum_{i=1}^{2} \phi_{1}^{*}(i) \psi(i) = (1/\sqrt{2}) \times 1 + (-i/\sqrt{2}) \times 0
\]

which yields \( c_1 = 1/\sqrt{2} \).

Find \( c_2 \). It is obtained by multiplying each component of \( \phi_{2}^{*} \) (note the subscript 2 since we are asking about \( c_2 \)) by the corresponding component of given wave vector \( \psi \) and sum over all \( i \):

\[
c_1 = \sum_{i=1}^{2} \phi_{2}^{*}(i) \psi(i) = (1/\sqrt{2}) \times 1 + (i/\sqrt{2}) \times 0
\]

which yields \( c_2 = 1/\sqrt{2} \).