Consider the coupled ordinary differential equations

\begin{align*}
\dot{x}_1 &= -2x_1(t) + x_2(t) \quad (1) \\
\dot{x}_2 &= x_1(t) - 2x_2(t) \quad (2)
\end{align*}

where the dot denotes differentiation with respect to time \( t \). We wish to solve these equations given initial conditions \( x_1(0) \) and \( x_2(0) \) at \( t = 0 \). We will solve this by a simple trick: adding and subtracting the two equations yields

\begin{align*}
\frac{d(x_1 + x_2)}{dt} &= -(x_1 + x_2) \quad (3) \\
\frac{d(x_1 - x_2)}{dt} &= -3(x_1 - x_2) \quad (4)
\end{align*}

We define the variables

\begin{align*}
y_+(t) &\equiv x_1(t) + x_2(t) \quad (5) \\
y_-(t) &\equiv x_1(t) - x_2(t). \quad (6)
\end{align*}

Substituting, we obtain

\begin{align*}
\dot{y}_+ &= -y_+ \quad (7) \\
\dot{y}_- &= -3y_- \quad (8)
\end{align*}

The linear combinations decouple the variables rendering them easy to solve. We obtain

\begin{align*}
y_+(t) &= y_+(0) e^{-t} \quad (9) \\
y_-(t) &= y_-(0) e^{-3t}. \quad (10)
\end{align*}

You should be crystal clear\(^1\) about solving the ode \( \dot{x} = -ax \) and how the initial condition enters the solution.

Let us write down the solution in terms of the original variables for completeness. Given \( y_+ \) and \( y_- \) (Equations (5) and (6)) you can check that

\begin{align*}
x_1(t) &= \frac{1}{2} (y_+(t) + y_-(t)) \quad (11) \\
x_2(t) &= \frac{1}{2} (y_+(t) - y_-(t)). \quad (12)
\end{align*}

\(^{1}\)“crystal clear” is a synonym for “you should be able to do it on an exam”
Note that these two equations are true at all times including the initial time $t = 0$. Thus we have

$$x_1(t) = \frac{1}{2} \left[ (x_1(0) + x_2(0)) e^{-t} + (x_1(0) - x_2(0)) e^{-3t} \right]. \quad (13)$$

Please write down an analogous expression for $x_2(t)$. This yields the time dependence of the two variables for the given initial conditions.

Let us write the given equations in matrix form:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (14)$$

We define

$$\frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}.$$ 

We can define the two ODEs compactly as

$$\frac{d}{dt} \vec{x} = M \vec{x}$$

where $M$ is the $2 \times 2$ matrix in Equation (14).

Find the eigenvalues of $M$. You should know how to find eigenvalues of $2 \times 2$ matrices quickly and correctly. They are $-1$ and $-3$. Note that the exponential time dependencies in $y_+$ and $y_-$ contain exactly these eigenvalues $-1$ and $-3$. This is, of course, not a coincidence. Furthermore let us find the corresponding eigenvectors. You should be able to do this. The normalized eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$ 

Note that the elements correspond exactly to the linear combinations that occur in $y_+$ and $y_-.$

Note that $M$ is a symmetric matrix. Write down three important properties of symmetric matrices and commit them to memory.

\[\text{this is another synonym for .....}\]
The formal way to proceed that you should be familiar with is to make a guess (\textit{ansatz}) for the \textbf{normal modes} in which all the dynamical degrees of freedom have the same time dependence. We assume

\begin{align*}
x_1(t) &= A_1 e^{\lambda t} \\
x_2(t) &= A_2 e^{\lambda t}.
\end{align*}

Both $x_1$ and $x_2$ are assumed to have the same exponential time dependence (since the equation is real and linear this is obvious from the one-variable case) but different amplitudes. Substituting into Equations 1 and 2 and canceling the exponential time dependence yields

$$\lambda \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = M \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$ 

Thus the time dependence is determined by the eigenvalues of $M$. The amplitudes are determined by the corresponding eigenvectors up to an overall constant.

If we denote the two eigenvalues of $M$ by $\lambda_1$ and $\lambda_2$ and the corresponding normalized eigenvectors by $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ we can write the solution as a linear combination of the two solutions with coefficients:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} e^{\lambda_2 t}.$$ 

\textit{Just to be sure substitute the above expression into the ODEs and check that they are satisfied.} Once more with emphasis the coefficients are determined by the initial conditions. Setting $t = 0$ in the above equation we obtain coupled algebraic equations that can be solved. An easier way to do this is to remember the properties of symmetric matrices: \textbf{The eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are orthogonal.} We will study this again later.