

1 Vector Analysis

We will consider scalar and vector functions(fields) which depend on three variables, x , y , and z the Cartesian components of a point in three-dimensional space: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. The scalar and vector fields will be typically denoted by $f(\vec{r})$ and $\vec{v}(\vec{r})$ respectively.

This course will deal with one of the fundamental concepts of physics, *fields*. For the present, a *field* is a physical quantity which takes on different values at different points in space. We will review the calculus of scalar and vector fields.

Recall Taylor's series for single-variable function, $f(x)$ around x_0 , i.e., imagine x is near x_0 so that $x - x_0$ is small:

$$f(x) \approx f(x_0) + \left. \frac{df}{dx} \right|_{x_0} (x - x_0) + \dots$$

Since the derivative of $f(x)$ at x_0 is the slope of the curve $y = f(x)$ at that point the expansion to first order represents a linear approximation to the curve at $x = x_0$ using the tangent to the curve at x_0 .

Next we consider the *directional derivative* of a scalar function $f(x, y, z)$.

Let us apply the Taylor series expansion in three variables to the scalar function $f(x, y, z) = f(\vec{r})$; we relate the value of f at $\vec{r}_0 + d\vec{\ell}$ to its value at \vec{r}_0 where $d\vec{\ell}$ is the infinitesimal vector,

$$d\vec{\ell} = dx\hat{x} + dy\hat{y} + dz\hat{z} ;$$

this yields

$$f(\vec{r}_0 + d\vec{\ell}) \equiv f(x_0 + dx, y_0 + dy, z_0 + dz) \approx f(\vec{r}) + dx \left(\frac{\partial f}{\partial x} \right)_{\vec{r}_0} + dy \left(\frac{\partial f}{\partial y} \right)_{\vec{r}_0} + dz \left(\frac{\partial f}{\partial z} \right)_{\vec{r}_0}$$

which can then be rewritten as

$$f(\vec{r} + d\vec{\ell}) \approx f(\vec{r}_0) + d\vec{\ell} \cdot \vec{\nabla} f \Big|_{\vec{r}_0}$$

where we have defined

$$\vec{\nabla} f \equiv \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z} \quad (1)$$

Another way of thinking about this is to note that the derivative of $f(\vec{r})$ along a direction specified by the unit vector \hat{n} is given by $\hat{n} \cdot \vec{\nabla} f$ evaluated at the point under consideration.

Example: Let $f(x, y, z) = x^2 e^{-yz}$ What is the directional derivative along $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, say at $(2, 0, 0)$ and $(2, 1, 0)$?

In addition to the definition of the gradient, the Taylor series expansion to first order written in terms of gradients you must remember the following results:

(i) Once more with emphasis, $\hat{n} \cdot \vec{\nabla} f$ yields the derivative of a scalar function f along the direction \hat{n} . Note that if you wish to determine the rate of variation along an arbitrary direction at a given point it suffices to know the rate along three orthogonal directions (the coordinate

axes); this information is compactly given by the gradient.

(ii) When $\nabla f(\vec{r})$ is non-zero (at a point) then it points in the direction along which f is increasing the fastest at that point.¹

(iii) The gradient of a function $\vec{\nabla}f$ is also orthogonal to the level surfaces of f (a level surface is one on which f is a constant.)

It is useful to think of $\vec{\nabla}$ as a vector operator; we define it in Cartesian coordinates to be

$$\hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z.$$

Examples in physics: The force per unit volume in a fluid is given by $-\vec{\nabla}p$ where p is the pressure which is a function of \vec{r} . Any small volume element of a fluid is subject to a force due to the difference in pressures on opposite faces of the volume element.

Let $\vec{h}(\vec{r})$ denote the amount of thermal energy that flows per unit area per unit time through an infinitesimal surface element perpendicular to the direction of the flow at the point \vec{r} . *Fourier's Law* of heat conduction states that \vec{h} is proportional to the gradient in the temperature T :

$$\vec{h} = -\kappa\vec{\nabla}T$$

where κ denotes the thermal conductivity. κ is a material-dependent constant, different for steel, copper or air. Why is there a negative sign? Heat flows downhill in the temperature.

Fick's Law of diffusion relates the mass current to gradients in the concentration or density while *Ohm's law* relates charge current density to gradients in the electric potential V :

$$\vec{j} = -\sigma\vec{\nabla}V.$$

These three are examples of phenomenological laws. They hold when the gradients are *small*. If the gradients are large the laws fail: For example, in using Hooke's Law the extension of a spring is proportional to the force for small extensions only; the spring will break if the force is increased beyond a limit.

You should make an analogous summary for divergence and curl.

¹The demonstration is obvious since the variation along any direction \hat{n} is proportional to $\hat{n} \cdot \vec{\nabla}f = |\nabla f| \cos \theta$ where θ is the angle between \hat{n} and the direction of $\vec{\nabla}f$. Clearly, this term is maximum when $\theta = 0$, i.e., \hat{n} and $\vec{\nabla}f$ are parallel.