Angular Momentum

We will begin our discussion of angular momentum by first considering an extended object with moment of inertia I, and rotating with angular velocity ω.

It turns out that angular momentum can be defined for a single particle relative to an axis - even a particle that is not rotating! But for now we will not consider this.

We know that in translational motion, we can define the linear momentum of an object as:

$$\vec{p} = m\vec{v}$$

We also know that we can relate the net external force on an object to its change in momentum:

$$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

This last relationship led to the Law of Conservation of linear momentum: If the net external force on an object is zero, then the momentum of the system (or object) is conserved - it does not change with time:

$$\text{if} \quad \sum \vec{F}_{ext} = 0 \qquad \text{then} \quad \vec{P}_i = \vec{P}_f$$

Angular Momentum

We know that the rotational analogues of mass m, force F, and velocity v are the moment of inertia I, the torque τ, and the angular velocity ω. So we can define the following quantities:

The angular momentum of an object with moment of inertia I and angular velocity ω:

$$\vec{L} = I\vec{\omega}$$

We use the symbol L for angular momentum. Note that it looks like linear momentum: (mass term)(velocity term). Note that the units of L are kg m^2/s.
Also note that **L is a vector**, and that it points in the **same direction as ω**.

We can relate the **net external torque** on an object to **its change in angular momentum**:

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

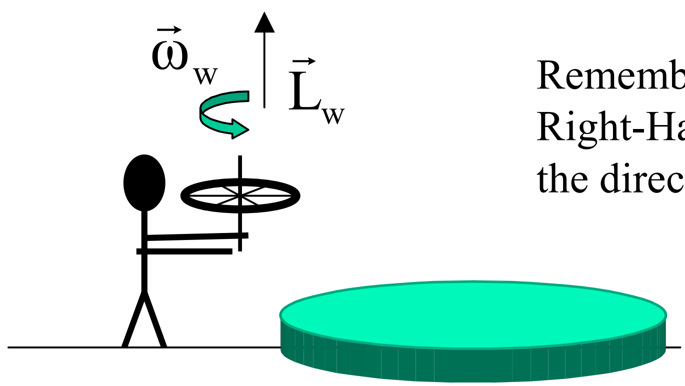
This last relationship leads to the **Law of Conservation of Angular Momentum**: If the net external torque on an object is zero, then the angular momentum of the system (or object) is conserved - it does not change with time:

$$\text{if} \quad \sum \vec{\tau}_{ext} = 0 \qquad \text{then} \quad \vec{L}_i = \vec{L}_f$$

Demo: Professor Spinning Out of Control!

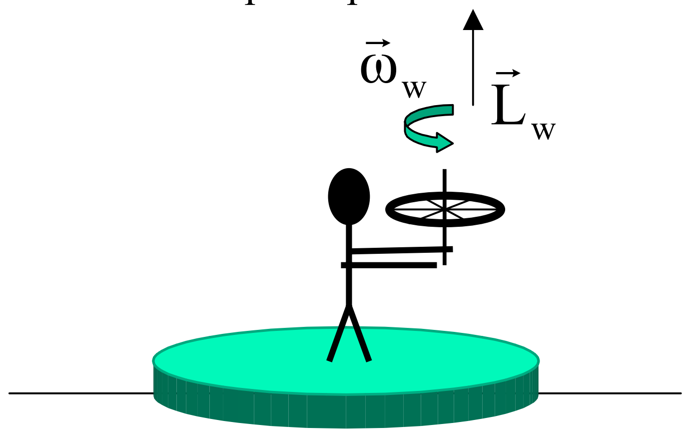
A professor holds a wheel that is spinning about its axis, and initially the axis is pointing straight up, as shown below. The professor then carefully steps onto a platform that is free to rotate about its axis. What happens - and why - when the professor flips the axis of the wheel 180°?

Before Prof steps on platform:

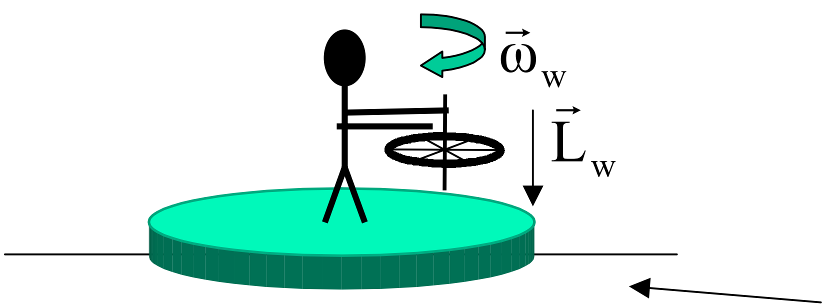


Remember we used the Right-Hand-Rule to find the direction of ω.

After Prof steps on platform:



After Prof flips wheel:



Let's consider our system as the Prof+wheel+platform. Then, when the Prof flips the wheel, no external torques are applied to the system. If this is true, then the angular momentum of the system must remain constant.

This picture is not quite finished!

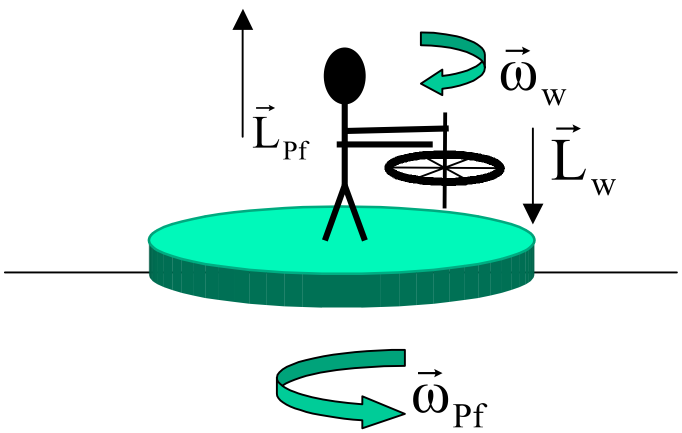
Demo: Professor Spinning Out of Control!

When we say that angular momentum is conserved - we mean it is conserved in both magnitude and direction. Initially, the angular momentum of the system is just that of the spinning wheel. It has magnitude L_W and its direction is upwards. When the professor is standing on the platform and flips the wheel, the magnitude of the angular momentum of the wheel remains L_W but its direction is now downwards. But the magnitude and direction of the system of Prof+platform+wheel must remain constant. This means that the Prof+platform must aquire an amount of angular momentum so that the system has angular momentum with magnitude L_W direction is upwards. To see what this means, let's define upwards as positive, then let:

$$L_W = \text{the angular momentum of the wheel}$$
$$L_{Pi} = \text{the initial ang mom of the Prof + Platform} = 0$$
$$L_{Pf} = \text{the final ang mom of the Prof + Platform}$$

Then:

$$\vec{L}_i = \vec{L}_f$$
$$L_W + L_{Pi} = -L_W + L_{Pf}$$
$$L_W + 0 = -L_W + L_{Pf}$$
$$L_{Pf} = 2L_W$$



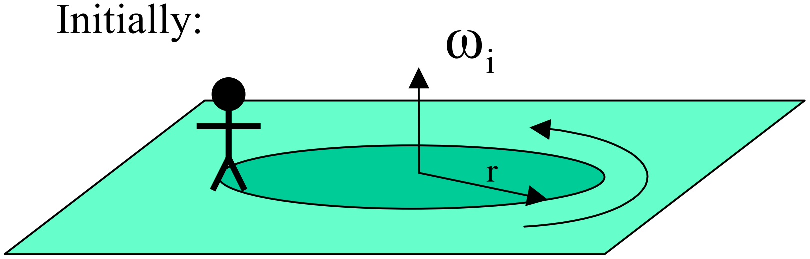
Example: Merry-Go-Round

A child with mass 25kg stands at the edge of a merry-go-round with mass 100kg and radius 2.0m. Initially the merry-go-round is spinning with angular velocity 2.5 rad/s. The child then walks toward the center of the merry-go-round (m-g-r).
A) What is the angular velocity of the system when the child is 1.0 m from the center?
B) What is the change in kinetic energy of the system?

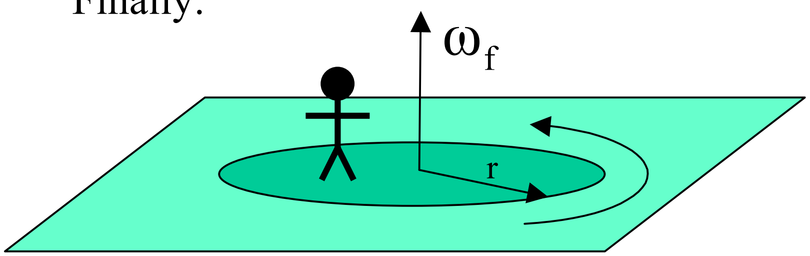
Since there are no external torques on the system, we know that:

$$L_i = L_f$$
$$I_i\omega_i = I_f\omega_f$$

Initially:



Finally:



The moment of inertia of the system is the sum of the moment of inertia of the m-g-r and the child:

$$I = I_c + I_m = m_c R_c^2 + \frac{1}{2} m_m R_m^2$$

Initially:

$$I_i = (25\text{kg})(2.0\text{m})^2 + \frac{1}{2}(100\text{kg})(2.0\text{m})^2 = 300 \text{ kg m}^2$$

Finally:

$$I_f = (25\text{kg})(1.0\text{m})^2 + \frac{1}{2}(100\text{kg})(2.0\text{m})^2 = 225 \text{ kg m}^2$$

Example: Merry-Go-Round Continued

We can now solve for the final angular velocity:

$$\omega_f = \frac{I_i \omega_i}{I_f} = 3.3 \text{ rad/s}$$

Important to keep in mind: as the child moves about on the merry-go-round, the angular momentum of the system remains constant.

B) Initial Kinetic energy of the system: $K_i = \frac{1}{2} I_i \omega_i^2 = 938 \text{ J}$

Final Kinetic energy of the system: $K_f = \frac{1}{2} I_f \omega_f^2 = 1250 \text{ J}$

How did the kinetic energy of the system increase? By walking in toward the center of the m-g-r, the child had to exert a force, and as a result did positive work. As a result the kinetic energy of the system increased. Note that if the child walks back toward the outside of the m-g-r, the kinetic energy of the system will decrease.