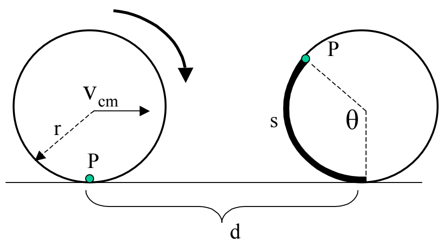
Rolling without Slipping

We will consider a particular kind of motion called rolling without slipping. The object rolling could be a disk, a hoop, a sphere, a cylinder, etc, but the important feature of this motion is that the object is not skidding or slipping over the surface. One feature of this motion that is very important is that it includes both translational and rotational motion

In this picture below, we have a wheel of radius r which is rolling without slipping. The center of mass of the wheel has a velocity v_{cm}, and the wheel is rotating with angular velocity ω. We will show that there is a relationship between these two velocities. Look at point P on the edge of the wheel, initially in contact with the surface. After a time, the wheel has travelled a distance d. In this time the wheel has rotated an angle θ, and point P is now at the place shown in the second figure.



You should be able to see that:

$$d = s$$
$$d = r\theta$$

If we differentiate these with respect to time, we find:

$$\frac{d}{dt}(d) = \frac{d}{dt}(r\theta)$$
$$v_{cm} = r\omega$$

Rolling without Slipping

So we have been able to relate the velocity of **the center of mass** to the angular velocity, and just to repeat our result, we find:

$$v_{cm} = r\omega$$

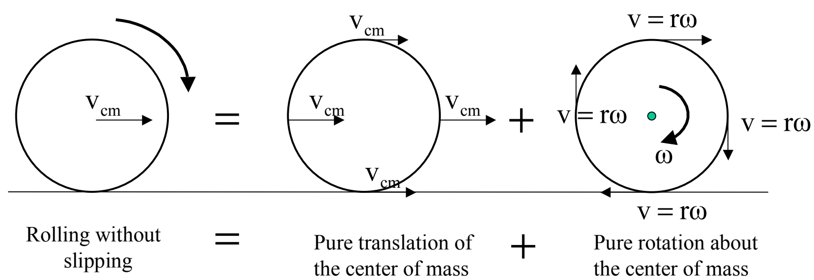
Differentiating this one more time, we find a relationship between the linear acceleration of the center of mass, and the angular acceleration:

$$\frac{d}{dt}(v_{cm}) = \frac{d}{dt}(r\omega)$$

$$a_{cm} = r\alpha$$

Rolling=Translation+Rotation

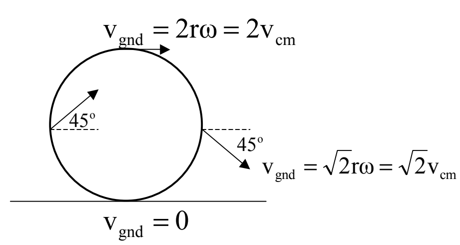
We can view rolling without slipping another way - namely as a combination of pure translation of the center of mass, plus pure rotation about the center of mass:



One way to see this is to imagine that you were running along side the wheel at the same speed as the wheel's center of mass - so that as far as you are concerned, the center of the wheel is at rest with respect to you. You would see the wheel rotating with angular speed ω. You would also see that each point on the rim of the wheel had a tangential velocity as in the last picture above, $v = r\omega$. But from that last page, we found that the velocity of the center of mass was $v_{cm} = r\omega$.

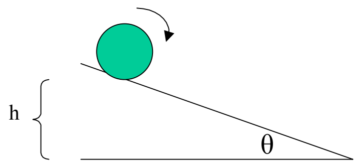
Rolling

Now let's examine the wheel from a point with respect to the ground. We want to determine the speed of points on the rim of the wheel with respect to the ground. To do this, we need to add (vectorially) the velocities of the last two pictures on the previous page, and we need to use the fact that $v = r\omega$ and $v_{cm} = r\omega$.



We find the following:
1) A point at the top of the wheel has a velocity (relative to the ground) which has a magnitude which is twice that of the center of mass.
2) The point which is in contact with the ground has a velocity which is zero. This is surprising, but will turn out to be very important when we look at the mechanical energy of a rolling object.

Example: Object Rolling Down an Incline

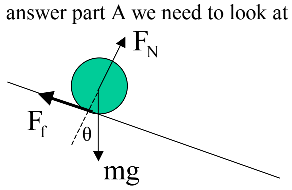


A round object (it could be a cylinder, sphere, disk, hoop) starts from rest and rolls without slipping down a incline. The object has a radius R and a mass M, and the incline makes an angle θ with the horizontal. The surface is rough, and the coefficient of static friction between the wheel and the surface is μ_s.

A) What is the acceleration of the object?

B) What is the speed of the object (the speed of its center of mass) after it a fallen a vertical distance h?

To answer part A we need to look at the forces and torques that act on this object:



$$\sum F_x = ma_x$$
$$-F_f + mg\sin\theta = ma_x$$

$$\sum F_y = ma_y = 0$$
$$F_N - mg\cos\theta = 0$$
$$F_N = mg\cos\theta$$

Why did the problem statement tell us the coefficient of static friction instead of kinetic friction - the object is moving! Remember, the point which is in contact with the surface is at rest - as long as there is no slipping - so static friction is present. If there was slipping, then kinetic friction would be present.

Example: Rolling (Continued)

However, note that the only thing we can say about friction is that: $F_f \le \mu_s F_N$

We can't use the equality - since we don't know that the object is on the verge of slipping…ask me about this if you don't understand this point.

There is one more thing we can use - Newton's 2nd law for rotational motion:

$$\sum \tau = I\alpha$$

The moment of inertia I will depend on whether the object is a sphere, cylinder, hoop, etc.

The axis of rotation in this case is about the center of mass. In this case the axis is moving (translating) but we can still use the above equation.
The only force that contributes to the torque is the force of friction (why?):

$$-F_f R = I\alpha$$

Remember, counterclockwise is positive.

Finally, there is a relationship between the linear acceleration of the center of mass, and the angular acceleration of the object:

$$a_x = -R\alpha$$

The negative sign is because if the linear acceleration is positive, the object moves down the incline, and the rotation is clockwise.

Example: Rolling (Continued)

To get the acceleration of the object, we use the following three equations:

$$-F_f + mg\sin\theta = ma_x \qquad -F_f R = I\alpha \qquad a_x = -R\alpha$$

$$-F_f + mg\sin\theta = ma_x$$
$$\left(\frac{I\alpha}{R}\right) + mg\sin\theta = ma_x$$
$$\left(-\frac{Ia_x}{R^2}\right) + mg\sin\theta = ma_x$$

$$a_x = \frac{mg\sin\theta}{m + \left(\frac{I}{R^2}\right)}$$

Example: Rolling (Continued)

Now that we have this answer, how can we use it? Let's apply it to the different cases of a hoop, a sphere, and a cylinder. First, note:

$$I_{hoop} = mR^2 \qquad I_{sphere} = \frac{2}{5}mR^2 \qquad I_{cyl} = \frac{1}{2}mR^2$$

We can write each of these equations as: $I = \beta m R^2$

Where β equals 1, 1/2, and 2/5 for a hoop, a cylinder, and a sphere, respectively.

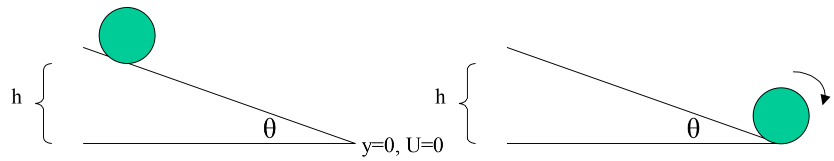
$$a_x = \frac{mg\sin\theta}{m + \left(\frac{\beta m R^2}{R^2}\right)}$$

$$a_x = \frac{g\sin\theta}{1+\beta}$$

So we find that the acceleration is largest for a sphere, and smallest for a hoop. The cylinder is in the middle.

Example: Rolling - Energy Method

Now let's answer part B), how fast is the object moving at the bottom of the incline. We could use the above result to find this, but let's instead use conservation of energy to find this.



Initial

$$U_i = mgh$$
$$K_i = 0$$

Final

$$U_f = mgy_f = 0$$
$$K_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Remember - the object is both translating and rotating - and there is kinetic energy associated with both!

Example: Rolling - Energy Method

We need to use the fact that: $v = R\omega$ and: $I = \beta m R^2$

Now the next question we need to ask, is mechanical energy conserved? The answer will be yes, if no nonconservative forces do any work. Now, we know that friction is present in this case - it is necessary if there is to be rolling in the first place. But, since the point of contact of the object is at rest relative to the surface - the force of friction does no work - there is no displacement! Think about this for a while. Also note - if there were slipping, then there would be kinetic friction, and mechanical energy would not be conserved.

Since mechanical energy is conserved in our case, we have:

$$E_i = E_f$$
$$K_i + U_i = K_f + U_f$$
$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\beta m R^2)\left(\frac{v}{R}\right)^2$$
$$mgh = \frac{1}{2}mv^2(1+\beta)$$
$$v = \sqrt{\frac{gh}{1+\beta}}$$

Remembering that β equals 1, 1/2, and 2/5 for a hoop, a cylinder, and a sphere, respectively, we find that at the bottom of the incline, the sphere is going fastest, the hoop the slowest, and the cylinder is again in the middle.

Example: Rolling - Energy Method

Things to keep in mind about the above results:

1) Both the acceleration and the speed are independent of the mass and the radius! They only depend upon the moment of inertia. This means that if we took two solid spheres of different mass and radii, and started them from rest at the top of the incline, they would both reach the bottom incline at the same time.
2) The acceleration in all cases (sphere, cylinder, hoop) is less than ($g \sin \theta$). This was the acceleration of a block down an frictionless incline.
3) One way to think of the results is to say that for objects with larger moments of inertia, more energy goes into rotation, and less into translation. Therefore the object with the largest moment of inertia - the hoop - has the smallest translational kinetic energy - and therefore has the smallest speed at the bottom of the incline.