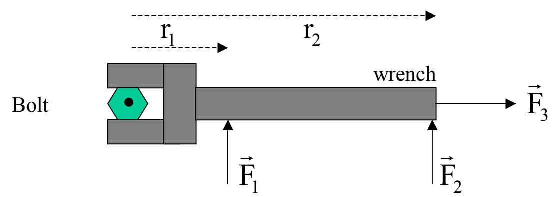
Dynamics of Rotational Motion

Let's imagine we have a rigid body, which is fixed to an axis about which it can rotate. Let's also imagine that this object is initally at rest. How do we get the object to rotate? We need to apply a force. However, there is more to the problem than this. Look at the situation below, where we have a bolt which we will try to turn with a wrench.



$$\left|\vec{F}_1\right| = \left|\vec{F}_2\right| = \left|\vec{F}_3\right|$$

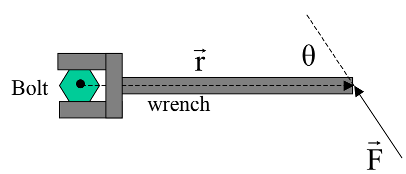
Which force will tend to rotate the bolt more easily? The force F_2. The distance of this force from the axis (also know as the moment arm) is larger than that of F_1. Also, note that the force F_3 will not tend to turn the bolt at all - this force acts along the axis.

We will define torque as a measure of the tendency of a force to cause or change rotation of a body about an axis. For the above three forces:

$$\tau_1 = r_1 F_1 \qquad \tau_2 = r_2 F_2 \qquad \tau_3 = 0$$

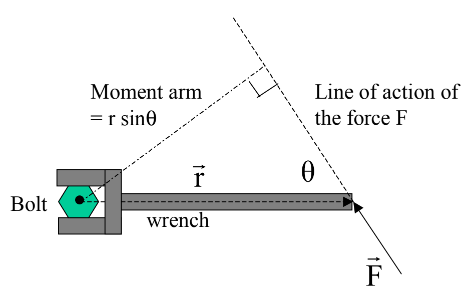
Torque

Consider the general application of a force to the above wrench:



$\vec{F}$ = the applied force
$\vec{r}$ = the vector distance from the axis to the point where the force is applied
θ = the angle between the applied force and the vector position

There are two ways of describing the torque of F.
1) Use the moment arm of the force F. To get the moment arm, extend F along its line of action (this just means the line along which it acts). Then find the perpendicular distance from the axis to this line. This perpendicular distance is called the moment arm.

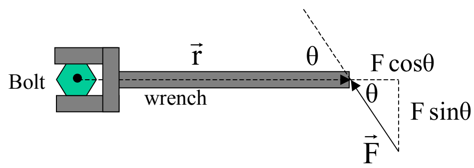


The torque is then just:

$$\tau = (\text{moment arm})\,(\text{force})$$
$$\tau = (r\ \sin\theta)F$$
$$\tau = rF\sin\theta$$

Torque

2) The second method of calculating the torque is to break F into components: one which is radial (along the direction of r - which therefore exerts no torque), and one which is perpendicular or tangential to r (which exerts all of the torque):



The torque is then just:

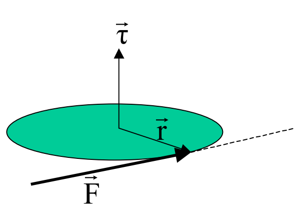
$$\tau = (\text{radial distance})(\text{perpendicular force})$$
$$\tau = r(F\sin\theta)$$
$$\tau = rF\sin\theta$$

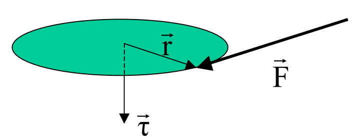
Of course we get the same answer for the torque in both cases.

Torque

Some notes on torque:

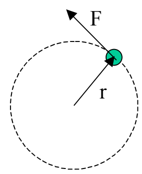
1) The units of torque are Newton-meters, Nm. You might recall that the units of work and energy are also Nm, which we called the Joule. These two quantities - energy and torque- are very different, and the units of torque are not Joules, they are Nm.
2) A force that tends to rotate an object counterclockwise yields a positive torque, while a force that tends to rotate an object clockwise yields a negative torque.
3) Torque is actually a vector - it is the result of the vector cross product of r and F, and its direction is given by the right hand rule: Let the fingers of your right hand point along the direction of r (from the axis to where the force is applied). Then curl your fingers in the direction of the force F (you may have to flip your hand upside down to do this). You thumb will then point in the direction of the vector torque.





Relation of Torque and Angular Acceleration

Imagine we had a single particle moving in a circle of radius r, under the influence of a tangential force F_t. Then we know:



1) From Newton's 2nd law: $F_t = ma_t$

2) The torque is just: $\tau = rF_t$

3) We can relate the tangential acceleration to the angular acceleration: $a_t = r\alpha$

Putting this all together we get: $\tau = rF_t = r(ma_t) = mr(r\alpha) = mr^2\alpha$

But mr^2 is just the moment of inertia of the mass m with respect to the origin, and so we have:

$$\tau = I\alpha$$

For many forces acting on an extended object, we can then write:

$$\sum \tau = I\alpha$$

This is Newton's 2nd Law for rotational motion.

Parallels between Translational and Rotational Motion

position	$x \Rightarrow \theta$
displacement	$\Delta x \Rightarrow \Delta\theta$
velocity	$v \Rightarrow \omega$
acceleration	$a \Rightarrow \alpha$
"Force"	$F \Rightarrow \tau$
"mass"	$m \Rightarrow I$

Motion under constant acceleration:

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v = v_0 + at \qquad \omega = \omega_0 + \alpha t$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Force and torque:

$$\sum F = ma \qquad \sum \tau = I\alpha$$

Work:

$$W = \int_{x_i}^{x_f} F(x)dx = F\Delta x \qquad W = \int_{\theta_i}^{\theta_f} \tau(\theta)d\theta = \tau\Delta\theta$$

Kinetic Energy:

$$K = \frac{1}{2}mv^2 \qquad K = \frac{1}{2}I\omega^2$$

Work-Kinetic Energy Theorem:

$$W = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$W = \Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

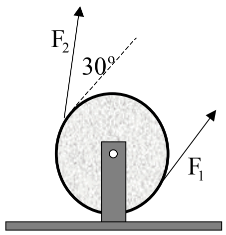
Power:

$$P = \frac{dW}{dt} = \frac{\vec{F}\cdot d\vec{x}}{dt} = \vec{F}\cdot\vec{v}$$

$$P = \frac{dW}{dt} = \tau\omega$$

Example: Torque and Angular Acceleration

A uniform disk with a mass of 24.3 kg and radius 0.314m rotates about its center. Two forces are applied as shown below.
A) What is the net torque? B) What is the resulting angular acceleration of the disk?



A) Define counterclockwise as + rotation:

$$\sum \tau = \tau_1 - \tau_2 = rF_1 - rF_2 \sin 30^0$$
$$\sum \tau = (0.314\text{m})(90.0\text{N}) - (0.314\text{m})(125.0\text{N})(0.5)$$
$$\sum \tau = +8.6 \text{ Nm}$$

B) We use Newton's 2nd Law for rotational motion:

$$\sum \tau = I\alpha$$

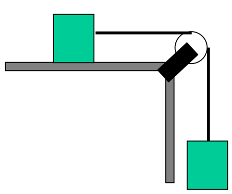
What is I for a disk?

$$I_{disk} = \frac{1}{2} mr^2 = \frac{1}{2}(24.3\text{kg})(0.314\text{m})^2$$
$$I_{disk} = 1.20 \text{ kg m}^2$$

$$\alpha = \frac{\sum \tau}{I} = \frac{8.6 \text{ Nm}}{1.2 \text{ kg m}^2} = +7.2 \text{ rad/s}^2$$

Example:Forces and Torques

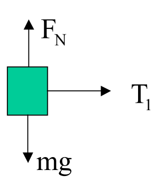
Two identical blocks each of mass m are connected by a very light string over a pulley of radius R and moment of inertia I. One block hangs freely while the other is on a frictionless surface. The blocks are released from rest, and the pulley does not slip (this is important). Find the tension in each part of the string, and the acceleration of the blocks.



To solve this problem, we need to first determine the forces and torques that act on each object, both blocks, and the pulley.

Also note: The tension in the string is different on either side of the pulley - because this is a real pulley, with mass, and moment of inertia - not an ideal pulley with neither.

Block 1:

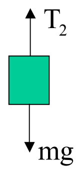


$$\sum F_x = ma_{1x}$$
$$T_1 = ma_{1x}$$

$$\sum F_y = ma_{1y} = 0$$
$$F_N - mg = 0$$
$$F_N = mg$$

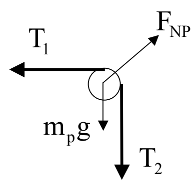
Example Continued

Block 2:



$$\sum F_y = ma_{2y}$$
$$T_2 - mg = ma_{2y}$$

Pulley:



$m_p g$ is the weight of the pulley

F_{NP} = the normal force the table exerts on the pulley

$$\sum \tau = I\alpha$$
$$T_1 r - T_2 r = I\alpha$$

Positive torques are those that tend to rotate the object counterclockwise.

Also note:

$$I = \frac{1}{2} m_p r^2$$

If $T_1 = T_2$ then $\alpha = 0$, which is the case for an ideal pulley.

Note that neither the weight nor the normal force exert a torque since their line of action goes through the axis.

Example Continued

To solve this problem, we need to realize that the three accelerations are related:

$$a_{1x} = -a_{2y} = -r\alpha$$

How did we get the relation for α? If the block on the surface has acceleration a, then the string that is in contact with the pulley is also has acceleration a. (Which is true only if the pulley does not slip - which is why we made a big deal about this in the statement of the problem. This is the tangential acceleration of a point on the rim of the pulley, and therefore we use the relation:

$$a_t = r\alpha$$

The minus sign in our equation comes from the fact that when the block on the surface has positive acceleration, then the pulley is rotating clockwise, which is negative angular acceleration.

$$T_1 = ma_{1x}$$

$$T_2 - mg = ma_{2y}$$
$$T_2 - mg = -ma_{1x}$$
$$T_2 = mg - ma_{1x}$$

$$T_1 r - T_2 r = I\alpha$$
$$T_1 r - T_2 r = \left(\frac{1}{2} m_p r^2\right)\left(\frac{-a_{1x}}{r}\right)$$
$$T_1 - T_2 = -\frac{1}{2} m_p a_{1x}$$

Substitute for T_1 and T_2 then solve for a_{1x}

Example Continued

$$T_1 - T_2 = -\frac{1}{2}m_p a_{1x}$$

$$ma_{1x} - (mg - ma_{1x}) = -\frac{1}{2}m_p a_{1x}$$

$$2ma_{1x} + \frac{1}{2}m_p a_{1x} = mg$$

$$a_{1x} = \frac{mg}{2m + \frac{1}{2}m_p}$$

Now we can substitute this back into the previous equations and solve for T_1 and T_2

$$T_1 = \frac{m^2 g}{2m + \frac{1}{2}m_p} \qquad T_2 = mg - \frac{m^2 g}{2m + \frac{1}{2}m_p}$$

What happens if the mass of the pulley is very small compared to the masses of the blocks?

$$a_{1x} \approx \frac{mg}{2m+0} = \frac{1}{2}g \qquad T_1 \approx \frac{m^2 g}{2m+0} = \frac{1}{2}mg \qquad T_2 \approx mg - \frac{m^2 g}{2m+0} = \frac{1}{2}mg$$