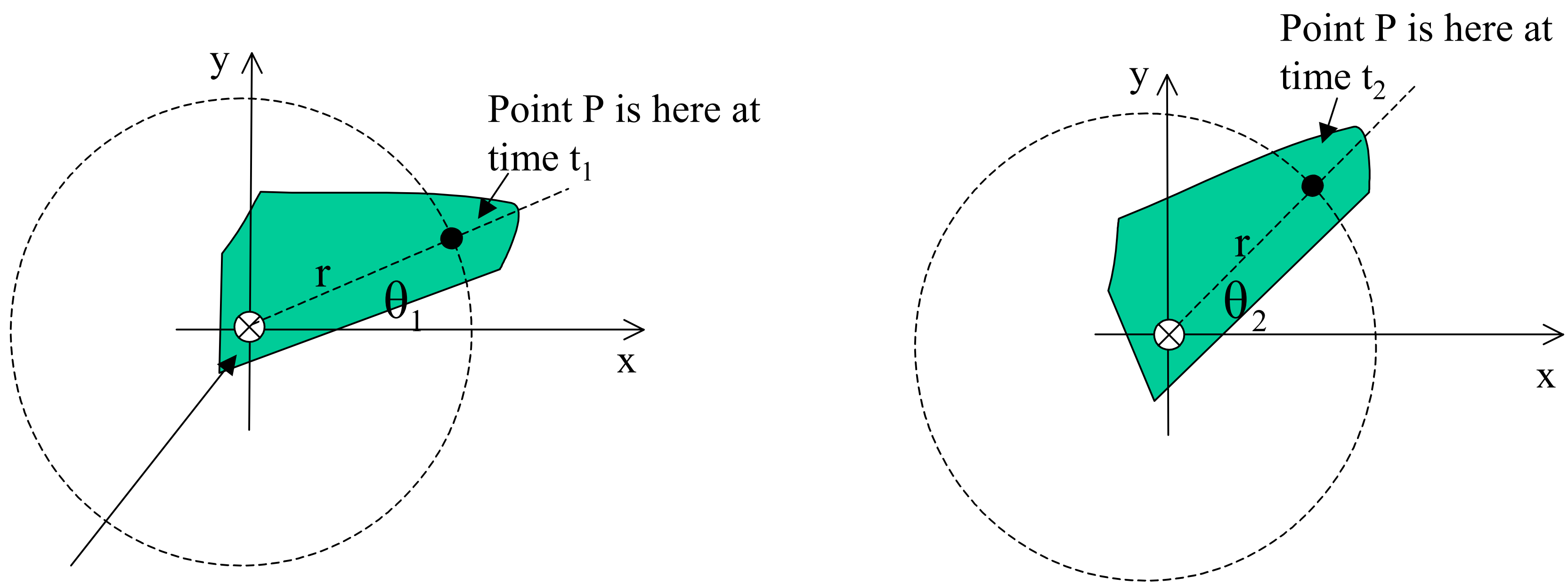
Rotation of a Rigid Body

Let's consider the motion of the object below. It is a rigid (non-deformable) object, which is fixed to a point about which it can freely rotate. Let's initially focus our attention on the point P, which is located on the object, at a distance **r** from the axis of rotation.



This object is fixed to this axis, and can only rotated around this axis.

At time t_1 the point P makes an angle θ_1 with the x-axis.
At some later time t_2 the point P makes an angle θ_2 with the x-axis.
We want to describe this motion. How do we do this?

The Kinematics of Rotational Motion

In describing translational motion, we used a number of quantities to help us describe the motion of an object in a straight line: position, displacement, velocity, and acceleration. We now will do the same thing for rotational motion:

1) **Angular position:** For this we use the angle θ, as measured from the x-axis. θ is measured in radians, not degrees, where 1 radian = 57.3 degrees. We define counterclockwise as the direction of increasing θ, and clockwise as the direction of decreasing θ.

2) **Angular displacement:** This is given by the change in angular position from one time to another. This is also measured in radians. $\Delta\theta = \theta_2 - \theta_1$
   A) Note that if an object makes one full revolution about the rotation axis, then its angular displacement would be 2π radians.
   B) The distance that the point P (located a distance r from the rotation axis) travels in going through an angular displacement $\Delta\theta$ is given by: $s = r\Delta\theta$
   This might be easier to see if you remember that if the point P goes through one complete revolution, it travels a distance equal to the circumference of a circle with radius r: $s = 2\pi\, r$
   C) If point P undergoes some angular displacement, every other point in the rigid body also goes through the same angular displacement. This might be easier to see if you imagine that the rigid object was a disk (like a CD) rotating about an axis through its center.

The Kinematics of Rotational Motion

3) **Angular velocity:** This is a measure of how fast the angular displacement is changing:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_2}{t_2 - t_1}$$

$$\omega = \frac{d\theta}{dt}$$

A) The units of angular velocity are in radians per second or rad/s.
B) The angular velocity is positive if the object is rotating in the direction of increasing θ (counterclockwise) and negative if the object is rotating in the direction of decreasing θ (clockwise).
C) Each point in the object has the same angular velocity.

4) **Angular acceleration:** This is a measure of how fast the angular velocity is changing:

Average: $$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_2}{t_2 - t_1}$$ Instantaneous: $$\alpha = \frac{d\omega}{dt}$$

A) The units of angular acceleration are rad/s^2.
B) Each point in the object has the same angular acceleration.

Motion under Constant Angular Acceleration

We have a number of parallels between translational motion and rotational motion:

position	$x \Rightarrow \theta$
displacement	$\Delta x \Rightarrow \Delta\theta$
velocity	$v \Rightarrow \omega$
acceleration	$a \Rightarrow \alpha$

In translational motion, we studied the special case of motion under constant acceleration:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$v = v_0 + at$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

We can do the same thing for motion under constant angular acceration:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = \omega_0^2 + 2\alpha\alpha(- \theta_0)$$

Where we need to identify the following quantities:

$\theta_0 =$

$\theta =$

$\omega_0 =$

$\omega =$

$\alpha =$

$t =$

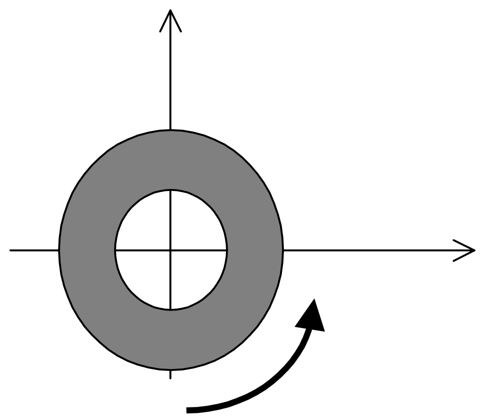
Example: Motion under Constant Angular Acceleration

A wheel rotates with constant angular acceleration of 3.5 rad/s^2. The angular velocity of the wheel is initially 2.0 rad/s.

A) What is the angular displacement after 2.0s?

B) What is the angular velocity after 2.0 sec?

Draw a simple picture:



Write what you know:

$$\theta_0 = 0$$
$$\theta = ?$$
$$\omega_0 = 2.0 \text{ rad/s}$$
$$\omega = ?$$
$$\alpha = 3.5 \text{ rad/s}^2$$
$$t = 2.0 \text{ s}$$

A)

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\theta = 0.0 + (2.0)(2.0) + \frac{1}{2}(3.5)(2)^2$$
$$\theta = 11 \text{ rad}$$

B)

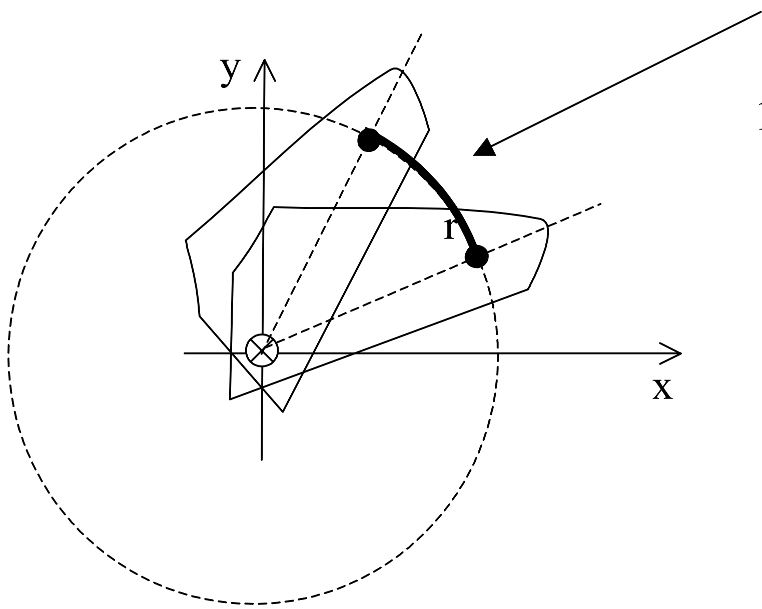
$$\omega = \omega_0 + \alpha t$$
$$\omega = 2.0 + (3.5)(2.0)$$
$$\omega = 9.0 \text{ rad/s}$$

Number of revolutions:

$$\theta = (11 \text{ rad})\frac{1 \text{ revolution}}{2\pi \text{ radians}} = 1.75 \text{ rev}$$

Relating Angular Variables to Translational Variables

Lets look back at the object we considered earlier. What we would like to do is to ask if there is any way to relate the angular position, angular velocity and angular acceleration to the more familiar ideas of translational position, velocity, and acceleration. Let's start by looking at the object when it has rotated through an angular displacement $\Delta\theta$ in a time Δt.



This arc length is: $s = r\Delta\theta$

1) Distance: Since the point P is located a distance, r from the axis, the distance it travels in the time Δt is:

$$s = r\Delta\theta$$

2) The average speed is just the distance traveled divided by time interval:

$$v_{avg} = \frac{r\Delta\theta}{\Delta t}$$

3) The instantaneous speed is obtained in the usual way:

$$v = \lim_{\Delta t \to 0} \frac{r\Delta\theta}{\Delta t} = r\frac{d\theta}{dt} = r\omega$$

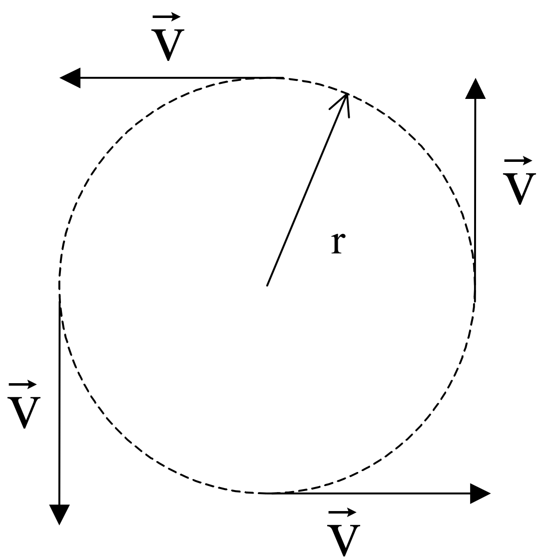
Relating Angular Variables to Translational Variables

4) If we ask how fast is the speed changing - that is, what is the acceleration, we find:

$$a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

The subscript t in the above equation is important. It means "tangential", that is, tangent to the circle that the point P is traveling in.

Earlier, we dealt with uniform circular motion, which is just a special case of rotational motion. In that case, we found that if an object was rotating in a circle, at constant speed, it had a centripetal (center-seeking), or radial (directed along the radius), acceleration:



$$a = a_c = a_r = \frac{v^2}{r}$$

For the special case of uniform circular motion, since the speed is constant, the tangential acceleration is zero. General rotational motion, however, has **both radial and tangential acceleration.**

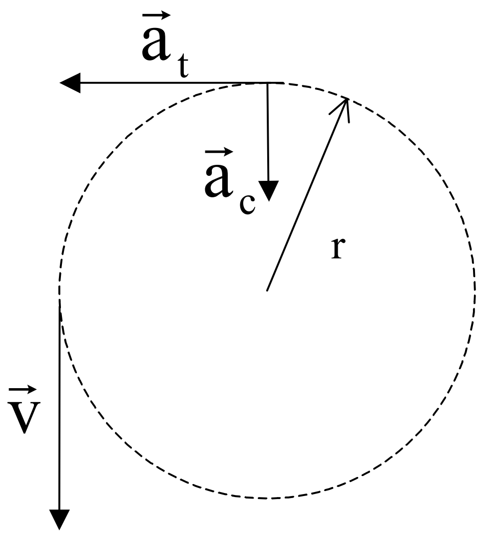
General Rotational Motion vs Uniform Circular Motion

General Rotational Motion:

$$v = r\omega$$

$$a_c = \frac{v^2}{r} = r\omega^2$$

$$a_t = r\alpha$$



Uniform Circular Motion:

$$v = r\omega \quad (v = \text{constant}, \omega = \text{constant})$$

$$a_c = \frac{v^2}{r} = r\omega^2$$

$$a_t = 0 \quad \left(\alpha = \frac{d\omega}{dt} = 0 \text{ if } \omega = \text{constant} \right)$$

Uniform circular motion has a linear acceleration due to the **changing direction** of the linear velocity vector.

General rotational motion **has this acceleration also**, but in addition has a linear acceleration due to the **changing magnitude** of the linear velocity vector.

Example: Rotation with Tangential Acceleration

A record turntable initially rotates at a rate of 33 rev/min, and takes 20s to come to rest.
A) What us the angular acceleration of the turntable, assuming the acceleration is uniform?
B) How many revolutions does the turntable make in coming to rest?
C) What are the magnitudes of the radial and tangential components of the linear acceleration of a point on the rim (r=14cm) at t=0?
D) What is the initial linear speed of a point on the rim?

First, what is the angular velocity, in radians/sec? $$\omega = 33\frac{\text{rev}}{\text{min}}\left(\frac{1\text{min}}{60\text{s}}\right)\left(\frac{2\pi\text{ rad}}{1\text{rev}}\right) = 3.46\text{ rad/s}$$

Next, write what you know:

$$\theta_0 = 0$$
$$\theta = ?$$
$$\omega_0 = 3.46\text{ rad/s}$$
$$\omega = 0$$
$$\alpha = ?$$
$$t = 20.0\text{ s}$$

A) Angular acceleration

$$\omega = \omega_0 + \alpha t$$
$$0 = 3.46 + \alpha(20.0)$$
$$\alpha = -0.173\text{ rad/s}^2$$

B) Number of revolutions

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\theta = 0.0 + (3.46)(20) + \frac{1}{2}(-0.173)(20)^2$$
$$\theta = 34.6\text{ rad}$$
$$\theta = 34.6\text{ rad}\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right) = 5.51\text{rev}$$

Example: Rotation with Tangential Acceleration (continued)

C) Radial and tangential components of the Linear acceleration (at t=0):

$$a_c = \frac{v^2}{r} = r\omega^2 = (0.14)(3.56)^2 = 1.68\ \text{m/s}^2$$

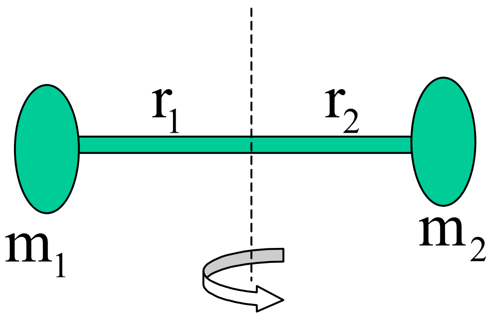
$$a_t = r\alpha = (0.14)(-0.173) = 0.0242\ \text{m/s}^2$$

D) Initial linear speed of a point on the rim:

$$v = r\omega = (0.14)(3.56) = 0.484\ \text{m/s}$$

Kinetic Energy of Rotation

Imagine that we have a rotating dumbbell - a light rod with a mass on each end. Let's assume that it is rotating about an axis through its center, as shown below. What would the kinetic energy of this object be?



We will assume that each mass is different, and that each mass is a different distance from the axis of rotation. If the dumbbell is rotating with angular velocity ω, then we know the two masses each have linear speed:

$$v_1 = \omega r_1$$
$$v_2 = \omega r_2$$

Given this information, we can now write down the kinetic energy of the system:

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

But let's now express this in terms of the angular velocity:

$$K = \frac{1}{2} m_1 (\omega r_1)^2 + \frac{1}{2} m_2 (\omega r_2)^2 = \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2\right)\omega^2$$

Kinetic Energy of Rotation

Now let's think about this last equation. If you look at how kinetic energy equation is written:

$$K = \frac{1}{2}(\text{mass term})(\text{speed term})^2$$

We know that in rotational motion, ω plays the role that speed does in linear motion. So in the last equation, we can think of the term in ( ) as the "mass" term:

$$K = \frac{1}{2}\underbrace{\left(m_1 r_1^2 + m_2 r_2^2\right)}\omega^2$$

Mass-like term. Called Moment of Inertia, I

Moment of Inertia

So the moment of inertia I plays the role of the mass m for rotating bodies. Generally, for a N objects rotating at a constant distance from an axis of rotation:

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots = \sum_{i=1}^{N} m_i r_i^2$$

The moment of inertia I:

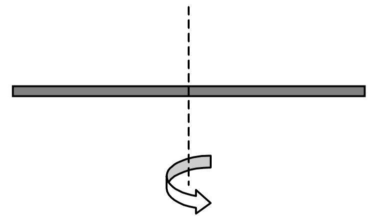
- It is constant for a rigid body, about a given axis of rotation. It tells how the mass of a rigid body is distributed with respect to that given axis.
- If the mass (or masses) are bigger, then I is bigger.
- If the distance from the axis is bigger, then I is bigger.
- I tells you how difficult it is to change the rotation of an object (like mass tells you how difficult it is to change the motion of an object).
- I is dependent upon the axis of rotation. If this changes, then I will change.

If you have a continuous distribution of mass, then the above discrete sum becomes an integral:

$$I = \int r^2 dm$$

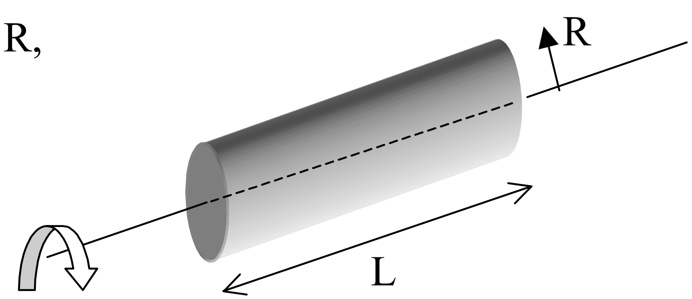
Moment of Inertia for Extended Objects

1) A thin rod of mass m and length L,
about an axis through its center:



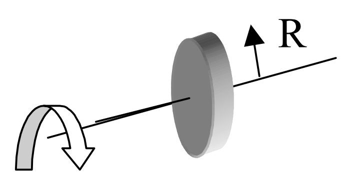
$$I = \frac{1}{12} m L^2$$

2) A solid cylinder (or thin disk) of mass m and radius R,
about its central axis:



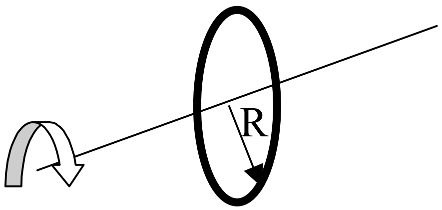
$$I = \frac{1}{2} m R^2$$

Note that the length of the cylinder
does not enter!



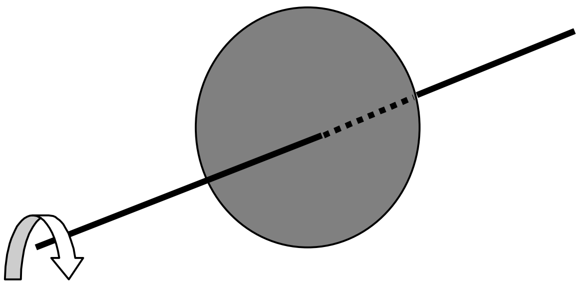
Moment of Inertia for Extended Objects

3) A thin hoop of mass m and radius R about its central axis:



$$I = m R^2$$

4) A solid sphere of mass m and radius R about any diameter:

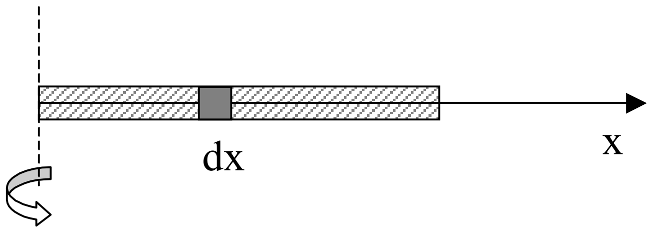


$$I = \frac{2}{5} m R^2$$

5) Guess: Will the moment of inertia of a spherical shell be larger or smaller than that of a solid sphere, of the same mass and radius?

Example: Calculation of the Moment of Inertia

Calculate the moment of inertia of a uniform thin rod of mass m, and length L, about one end, as shown below.



The mass of the shaded element of the rod dx is: $dm = \frac{m}{L} dx$

Think about why this is true.
As long as the rod is uniform,
the mass of 1/2 of the rod is m/2
the mass of 1/10 of the rod is m/10
the mass of a fraction dx/L of the rod is m(dx/L)

$$I = \int r^2 dm = \int_0^L x^2 dm = \int_0^L x^2 \left(\frac{m}{L}\right) dx = \left(\frac{m}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \left(\frac{m}{L}\right)\left(\frac{L^3}{3}\right) = \frac{1}{3} mL^2$$

The Parallel Axis Theorem

This theorem relates the moment of inertia of an object of mass M about **any axis that passes through the center of mass**, to any other **parallel axis** a distance d away:

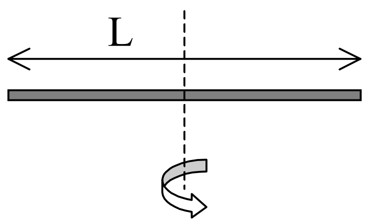
$$I = I_c + m\,d^2$$

I_c is the moment of inertia about a given axis that passes through the center of mass.

M is the mass of the object, and d is the distance between the two axes

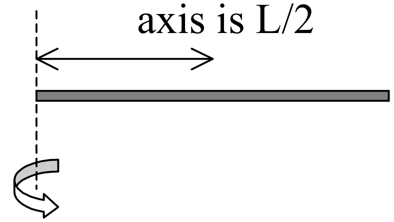
Can we apply this to the problem we just did - the moment of inertia of the thin rod about one end?

A thin rod of mass m and length L, about an axis through its center:



$$I_c = \frac{1}{12} m\,L^2$$

Distance from old axis is L/2



$$I = I_c + md^2$$

$$I = \frac{1}{12} m\,L^2 + m\left(\frac{L}{2}\right)^2$$

$$I = \frac{1}{3} m\,L^2$$

Addition of Moments of Inertia

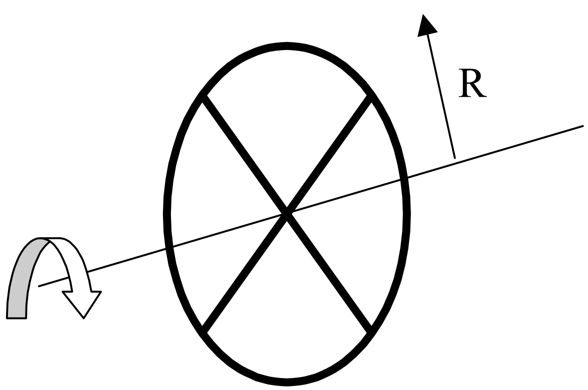
If we have an object that is made of 2 or more pieces, and this object is being rotated about some axis. If we know the moment of inertia of each piece separately about that axis, the the moment of inertai of the complete object is just the sum of the individual moments of inertia:

$$I_{tot} = I_1 + I_2 + I_3 + \ldots$$

Example: We have a wheel (of radius R and mass M_W) with two spokes (each of length 2R and mass M_s) which is rotated about an axis passing through its center. What is the moment of inertia of this object?

A) The wheel is like a thin hoop: $I_w = M_w R^2$

B) The spokes are like thin rods: $I_s = \frac{1}{12} M_s (2R)^2$



C) The total moment of inertia is the sum of the I's of the hoop and the two thin rods:

$$I_{tot} = M_w R^2 + 2\left[\frac{1}{12} M_s (2R)^2\right]$$

$$I_{tot} = M_w R^2 + \frac{2}{3} M_s R^2$$