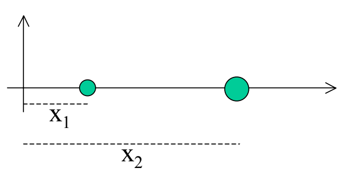
Center of Mass

Center of Mass:
- Up till now, we have dealt with single "particles"
- These were things like a box sliding down a plane, a person running, a ball thrown through the air
- We "idealized" the motion of these complicated objects
- But real objects are "extended bodies", and their motion is complicated:
  - Translation, rotation, vibration, etc.
- If a real object is subject to a net force, and undergoes translation and rotation, there us one point that moves in the same path that a single particle would, if subjected to the same net force: This point is the Center of Mass (CM).

The general motion of an object can be described as:
- Translation of the CM.
- Rotation about the CM.

Center of Mass

Center of Mass in 1 Dimension:



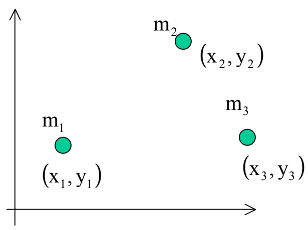
$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

For a system of N particles, we can generalize this formula to:

$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M} \sum_{i=1}^{N} m_i x_i$$

Center of Mass

To generalize to 2-dimensions:



$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

To generalize to 3-dimensions: $\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i$

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i x_i \qquad y_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i y_i \qquad z_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i z_i$$

Center of Mass for Rigid Bodies

For a rigid body - meaning a body made of a continuous (rather than discrete) colletion of particles, we need to replace the sum by an integral:

$$x_{cm} = \frac{1}{M}\int x\, dm \qquad y_{cm} = \frac{1}{M}\int y\, dm \qquad z_{cm} = \frac{1}{M}\int z\, dm$$

Newton's 2nd Law For Systems of Particles

Why is the CM so important? It tells us about the motion of a system of particles:

$$\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i$$

$$M\vec{r}_{cm} = \sum_{i=1}^{N} m_i \vec{r}_i$$

$$\frac{d}{dt}\left[M\vec{r}_{cm} = \sum_{i=1}^{N} m_i \vec{r}_i\right]$$

$$M\frac{d\vec{r}_{cm}}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt}$$

$$M\vec{v}_{cm} = \sum_{i=1}^{N} m_i v_i$$

$$\frac{d}{dt}\left[M\vec{v}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i\right]$$

$$M\frac{d\vec{v}_{cm}}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{v}_i}{dt}$$

$$M\vec{a}_{cm} = \sum_{i=1}^{N} m_i \vec{a}_i = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots$$

$$m_1\vec{a}_1 = \sum \vec{F}_1 \qquad m_2\vec{a}_2 = \sum \vec{F}_2$$

Newton's 2nd Law For Systems of Particles

$$M\vec{a}_{cm} = \sum F_1 + \sum F_2 + \ldots$$

Sum of **all** of the forces on particle 1 — Sum of **all** of the forces on particle 2

But some of the forces on m_1 come from m_2 and m_3 and….

But some of the forces on m_2 come from m_1 and m_3 and….

Internal forces = Forces of one particle of a system on another.
External forces = all the other forces on a system not due to particles inside the system.

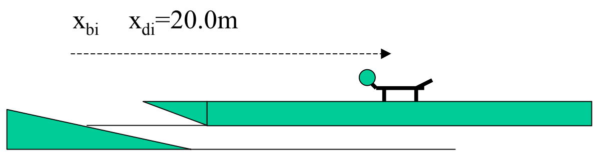
$$M\vec{a}_{cm} = \sum F_{ext} + \sum F_{int}$$

But by Newton's 3rd Law: $\sum F_{int} = 0$

$$M\vec{a}_{cm} = \sum F_{ext}$$

Example: Newton's 2nd Law for a System of Particles

Chapter 9, Problem 23: A dog with mass 10kg is standing on a flat boat so that he is 20m from the shore. He walks 8.0m on the boat to the shore and then halts. The boat has a mass of 40.0kg, and you can assume that there is no friction between the boat and the water. How far is the dog from the shore?



The initial position of the dog is x_{di}=20.0m.
The initial position of the boat is x_{bi}=20.0m.

The final position of the dog is $x_{df} = x_{bf} - 8.0$
The final position of the boat is x_{bf}.

Example: Newton's 2nd Law for a System of Particles

Since the net external force on the system of dog+boat is zero, then the acceleration of the system does not change.

Since the dog+boat system was initially at rest, then the system stays at rest, and therefore the center of mass of the system does not change.

$$x_{cmi} = x_{cmf}$$

$$\frac{m_b x_{bi} + m_d x_{di}}{m_b + m_d} = \frac{m_b x_{bf} + m_d x_{df}}{m_b + m_d}$$

$$40(20) + 10(20) = 40(x_{bf}) + 10(x_{bf} - 8.0)$$

$$800 + 200 = 50x_{bf} - 80$$

$$x_{bf} = \frac{1080}{50} = 21.6\text{m}$$

$$x_{df} = x_{bf} - 8.0 = 13.6\text{m}$$

Linear Momentum

Momentum:
- Important because in a number of situations it is conserved.
- Starting with momentum one can derive Newton's Laws
- Definition of momentum for a single particle:

$$\vec{p} = m\vec{v}$$

Note that:
- Momentum is a vector.
- It has the same direction as the velocity.
- It has units of (kg m / s).

Momentum and Newton's 2nd Law

One can express Newton's 2nd law using momentum (this is how Newton did it):

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt}$$

← Most general statement of Newton's 2nd Law, allows both mass and velocity to change with time

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

← Can only take this last step if the mass of the object does not vary with time (which is true for all the problems we have done so far).

Linear Momentum and Systems of Particles

We can define the momentum of a system of particles:

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots = \sum_{i=1}^{N} m_i v_i$$

But earlier we found using the center of mass equation that: $M\vec{v}_{cm} = \sum_{i=1}^{N} m_i v_i$

So we can relate the total momentum of a system of particles to the total mass M of the system and the velocity of the center of mass:

$$\vec{P} = M\vec{v}_{cm}$$

Conservation of Linear Momentum

Let's take the derivative of the total momentum with respect to time:

$$\frac{d\vec{P}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = M\frac{d\vec{v}_{cm}}{dt} = M\vec{a}_{cm}$$

But earlier we found using the center of mass equation that: $M\vec{a}_{cm} = \sum F_{ext}$

$$\sum F_{ext} = \frac{d\vec{P}}{dt}$$

But what happens if there is no external force?

if $\sum F_{ext} = 0$

then $\frac{d\vec{P}}{dt} = 0$

or

$\vec{P} =$ constant with time

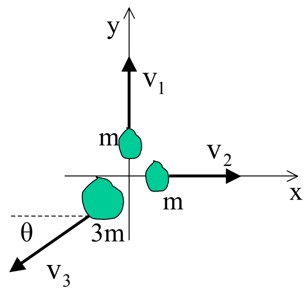
$\Delta\vec{P} = 0$

$\vec{P}_f = \vec{P}_i$

Example: Conservation of Linear Momentum

Chapter 9, Problem 43: A vessel at rest explodes, breaking into 3 pieces. Two pieces which have equal mass, fly off perpendicular to one another with the same speed of 30m/s. The third piece has 3 times the mass of each other piece. What are the direction and magnitude of its velocity immediately after the explosion?

I) Draw a picture of the problem



II) Is momentum conserved? If so, why?

Example: Conservation of Linear Momentum

II) Is momentum conserved? Yes! The only forces are internal.

$$\sum F_{ext} = 0 \qquad \vec{P}_f = \vec{P}_i$$

The initial momentum was zero (the vessel was at rest) so:

$$P_{xi} = 0 \qquad P_{yi} = 0$$

The final momentum is:

$$P_{xf} = m_2 v_2 - m_3 v_3 \cos\theta \qquad P_{yf} = m_1 v_1 - m_3 v_3 \sin\theta$$

Now apply conservation of momentum in the x and y directions:

$$P_{xf} = P_{xi}$$
$$m_2 v_2 - m_3 v_3 \cos\theta = 0$$
$$m_2 v_2 = m_3 v_3 \cos\theta \quad \text{eq(1)}$$

$$P_{yf} = P_{yi}$$
$$m_1 v_1 - m_3 v_3 \sin\theta = 0$$
$$m_1 v_1 = m_3 v_3 \sin\theta \quad \text{eq(2)}$$

Example Continued:

III) Divide eq(1) by eq(2):

$$\frac{m_1 v_1 = m_3 v_3 \sin\theta}{m_2 v_2 = m_3 v_3 \cos\theta}$$

$$\tan\theta = \frac{m_1 v_1}{m_2 v_2} = \frac{m\,(30\text{m/s})}{m\,(30\text{m/s})} = 1.0 \qquad \theta = 45^{\circ}$$

IV) Now substitute this value of θ into either eq(1) or eq(2):

$$m_2 v_2 = m_3 v_3 \cos\theta$$

$$m\,(30) = 3mv_3 \cos 45$$

$$v_3 = \frac{30}{3(0.707)} = 14 m/s$$

Impulse and Linear Momentum

We already know the relationship between force and change in momentum:

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Let's think about this slightly differently. Imagine that just one force acts on an object, and that it acts for a time Δt. We assume that the force is constant over this time Δt. We can express the above relationship like this:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{\Delta\vec{p}}{\Delta t}$$

If we now ask, what is the change in momentum of an object, if a force F acts on it for a time Δt:

$$\Delta\vec{p} = \vec{F}\Delta t$$

This change in momentum is given a special name Impulse, and we use the symbol J to represent it. Since impulse is just a change in momentum it has the same units a momentum.

$$J = \Delta\vec{p} = \vec{F}\Delta t$$

What if the force is not constant over the time interval?

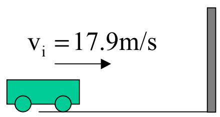
$$J = \Delta\vec{p} = \int_{t1}^{t2} \vec{F}(t)dt$$

Example: Impulse

A 70 kg person is a passenger in a car. The car is moving at 17.9m/s (~40mi/h) when it hits a concrete barrier. If the stopping time is 100ms (=0.1s), determine: A) the impulse that acts on the person, and B) the average force that the seat belt exerts on the person.

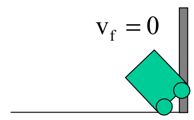
Inital:

$v_i = 17.9\text{m/s}$



Final:

$v_f = 0$



A)
$$J = \Delta\vec{p} = p_f - p_i$$
$$J = mv_f - mv_i$$
$$J = 0 - (70\text{kg})(17.9\text{m/s})$$
$$J = -1.3\times10^3\,\text{kg m/s}$$

How can the impulse be made smaller?
How can the force be made smaller?

B)
$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{\Delta\vec{p}}{\Delta t}$$
$$F = \frac{-1.3\times10^3}{0.1}$$
$$F = -1.3\times10^4\,\text{N}$$

This force is equivalent to the weight of about 1 ton.

Collisions

Notice that in the last problem, we ignored any other forces in the problem (like friction between the tires and the road). Give the size of the impulsive force in that case, we know that friction is much smaller that this. (Prove this to yourself. Assume the car was 1000kg, and that a typical coefficient of kinetic friction is 0.4. How big is the force of friction?).

We will now discuss what happens in detail when events of this type - Collisions -occur. In these collisions we will consider two case:
1) No external forces are present. OR
2) External forces are present, but they are very small compared to the impulsive forces present in the collision.

With these assumptions, we know that momentum is conserved in collisions

if $\sum F_{ext} = 0$

then $\frac{d\vec{P}}{dt} = 0$

→ $\vec{P} = \text{constant with time}$

or

→ $\Delta\vec{P} = 0$, $\vec{P}_f = \vec{P}_i$

Types of Collisions

There are basically two types of collisions:
A) Elastic: Both momentum and Kinetic energy are conserved:

$\vec{P}_f = \vec{P}_i$ $K_f = K_i$

B) Inelastic: Momentum IS conserved, but Kinetic energy is NOT.

$\vec{P}_f = \vec{P}_i$

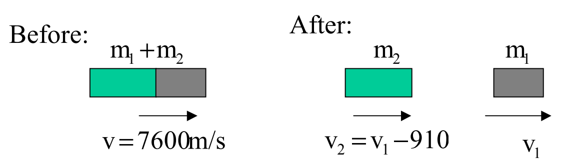
In the inelastic case, KE could be lost through heat, deformation, etc. Also, KE could be gained in an inelastic collision (for example, an explosion can be viewed as an inelastic collision in reverse).

NOTE: In <u>both</u> types of collisions, <u>momentum is conserved</u>. This is true as long as the net external force is zero (or very small).

Example: Momentum Conservation

Chapter 9, Prob 39: A rocket is traveling at a speed of 7600m/s. The rocket is made up of two parts that are clamped together: a rocket case that has a mass of 290.0kg and a payload that has a mass of 150.0kg. A clamp is released, and a compressed spring separates the two parts such that they have a relative velocity of 910.0m/s.

A) What are the speed of the two parts after separation?



After the clamp is released, m_1 is moving 910m/s faster than m_2.

Example: Momentum Conservation

Part A, Continued:

Since no external forces act, momentum is conserved.

Final momentum: $P_i = (m_1 + m_2)v = (290.0 + 150.0)\cdot 7600 = 3.34\times 10^6 \text{ kg m/s}$

Initial momentum: $P_f = m_1 v_1 + m_2 v_2 = 290(v_1 - 910) + 150 v_1 = 440 v_1 - 2.64\times 10^5$

$$P_i = P_f$$
$$3.34\times 10^6 = 440 v_1 - 2.64\times 10^5$$
$$v_1 = 8200\text{m/s}$$
$$v_2 = 7290\text{m/s}$$

Example: Continued

B) What is the kinetic energy before and after the separation?

$$K_i = \frac{1}{2}mv^2 \qquad K_f = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$
$$K_i = \frac{1}{2}(290+150)(7600)^2 \qquad K_f = \frac{1}{2}(290)(7290)^2 + \frac{1}{2}(150)(8200)^2$$
$$K_i = 1.271\times 10^{10}\text{ J} \qquad K_f = 1.275\times 10^{10}\text{ J}$$

So, even though momentum is conserved (because no external forces act), kinetic energy is not conserved. There is more kinetic energy after the separation than before. Where did this additional energy come from?

It came from the energy stored in the spring loaded clamp - the potential energy of the spring was converted to additional kinetic energy of the rocket parts.