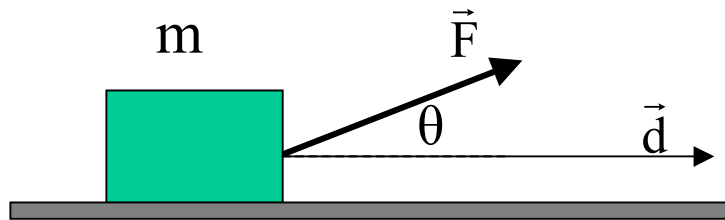


## Work and the Dot-Product

Earlier, we wrote down the expression for work in terms of the component of the force along the displacement, multiplied by the displacement:



$$W = (F \cos \theta) d$$

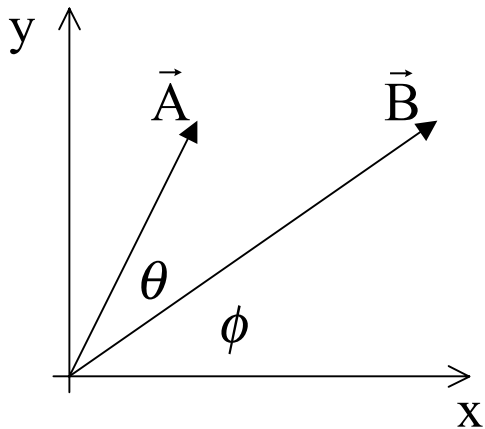
There is a more general way to write this, using the **Vector Dot-Product**:

$$W = \vec{F} \cdot \vec{d} \equiv |\mathbf{F}| |\mathbf{d}| \cos \theta$$

Where by  $\theta$  we mean the angle between the vector  $F$  and the vector  $d$ .

# The Vector Dot-Product

Consider two vectors, A and B. B makes an angle  $\phi$  with the x axis, while A makes an angle  $\phi+\theta$  with the x-axis (meaning the angle between A and B is  $\theta$ ). The vector dot product of A and B is:

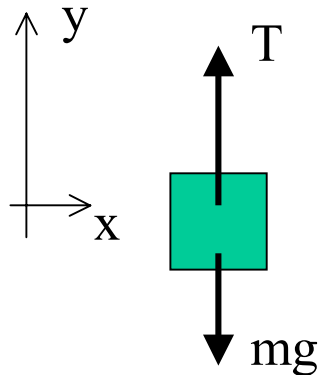


$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y \\ &= A \cos(\theta + \phi) B \cos(\phi) + A \sin(\theta + \phi) B \sin(\phi) \\ &= AB [\cos(\theta + \phi) \cos(\phi) + \sin(\theta + \phi) \sin(\phi)] \\ &= AB \cos \theta\end{aligned}$$

The result of the vector dot product is a scalar (a number with a magnitude) not another vector (which is a number with a magnitude and a direction).

## Example: Work done by weight

A 220kg load is lifted 21.0m vertically, with an acceleration  $a=0.150g$ , by a single cable.



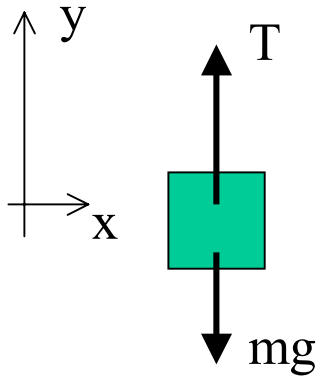
A) What is the net force on the object? Remember, by definition the net force comes from Newton's 2nd Law, and is just the mass of the object times its acceleration:

$$F_{\text{net}} = \sum F = ma$$

$$F_{\text{net}} = (220\text{kg})(0.15)(9.80\text{m/s}^2) = 323\text{N}$$

## Example: Work done by weight

B) What is the tension in the cable?



$$\sum F = ma$$

$$T - mg = ma$$

$$T = mg + ma = (220\text{kg})(9.80\text{m/s}^2) + 323\text{N}$$

$$T = 2.48 \times 10^3 \text{ N}$$

C) What is the net work done on the load?

$$W_{\text{net}} = \vec{F}_{\text{net}} \cdot \vec{d} = (323\text{N})(21.0\text{m}) = 6.79 \times 10^3 \text{ J}$$

CONTINUED =>

## Example(cont): Work done by weight

D) What is the work done by the cable on the load?

$$W_{\text{cable}} = \vec{T} \cdot \vec{d} = (2.48 \times 10^3 \text{ N})(21.0 \text{ m}) = 5.21 \times 10^4 \text{ J}$$

E) What is the work done by gravity on the load?

$$W_{\text{gravity}} = (m\vec{g}) \cdot \vec{d} = (-2.16 \times 10^3 \text{ N})(21.0 \text{ m}) = -4.53 \times 10^4 \text{ J}$$

NOTE:  $W_{\text{net}} = W_{\text{cable}} + W_{\text{gravity}}$

F) If the load started from rest, what is its final speed (after it has been lifted 21.0m)?

$$W_{\text{net}} = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

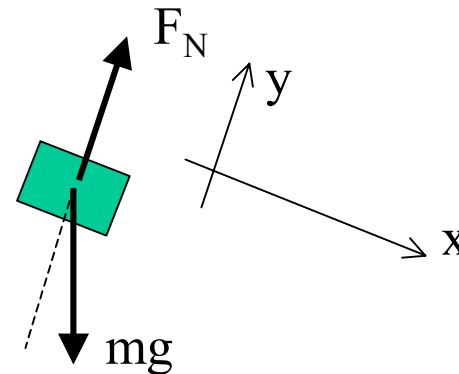
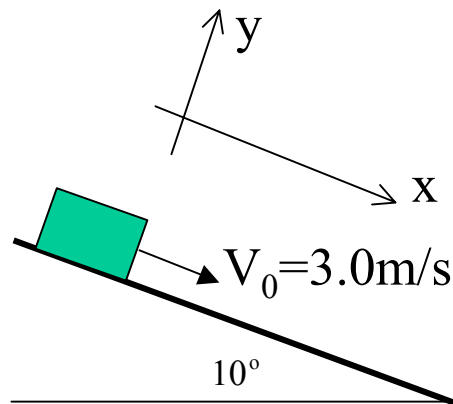
$$6.79 \times 10^3 \text{ J} = \frac{1}{2}(220 \text{ kg})v_f^2 + 0$$

$$v_f = \sqrt{\frac{2(6.79 \times 10^3 \text{ J})}{(220 \text{ kg})}} = 7.86 \text{ m/s}$$

## Example: Work and kinetic energy.

A 50kg box is at the top of an icy hill. It is initially moving down the hill with a speed of 3.0m/s.

What is its speed after it has gone 50m down the hill?



## Example Continued : Work and kinetic energy.

The box's displacement is along the x-axis as defined above. The normal force has no component along the x-axis, and therefore does no work. However the force of gravity does have a component along the x-axis. Let's relate this work done to the change in kinetic energy of the box:

$$W = \vec{F} \cdot \vec{d} = (mg)d \sin\theta$$

$$W = (50\text{kg})(9.80\text{m/s}^2)(50\text{m})(0.17)$$

$$W = 4254\text{J}$$

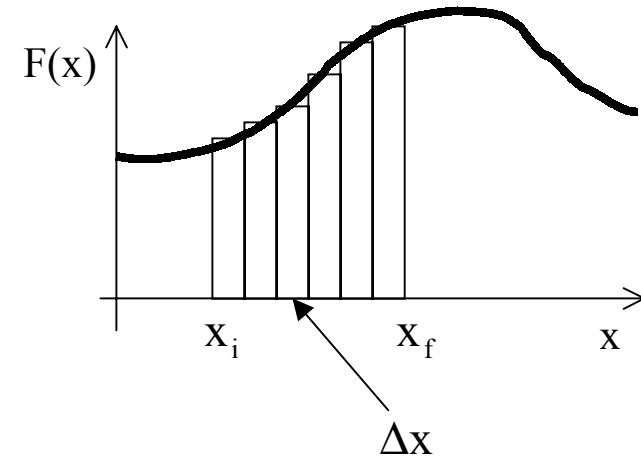
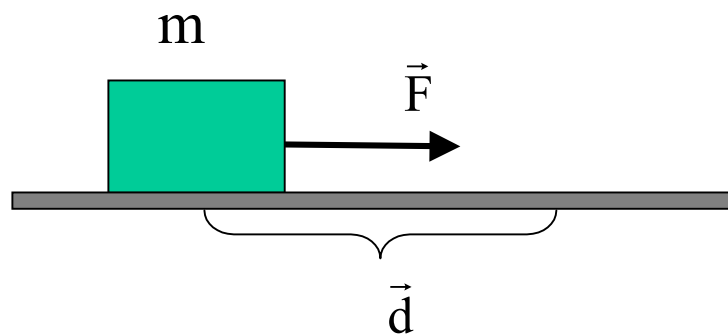
$$W = \Delta K = K_f - K_i$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$v_f = \sqrt{\frac{2W}{m} + v_i^2} = 13.4\text{m/s}$$

## Work Done by a Variable Force

Let's examine a simple situation in which  $F$  and the displacement are parallel. If  $F$  is constant all along the displacement, then the work done by the force, on the object, is  $W = Fd$ . What if  $F$  varies with position, as shown in the graph below? How would we calculate the work done in moving the object from position  $x_i$  to position  $x_f$ ?



We begin by dividing the path up into a small number of intervals  $\Delta x$ . In each of these very small intervals,  $F$  is **approximately constant**, so the work done is  $W = F \Delta x$ . To get the approximate work done over the full path, we need to add up all the small intervals. To get a better and better approximation of the work, we need to divide the path up into smaller and smaller intervals  $\Delta x$ . In the limit, when  $\Delta x$  goes to zero, **the true work done is the integral of  $F(x)$  over the path from position  $x_i$  to position  $x_f$ .**

$$W = \lim_{\Delta x \rightarrow 0} \sum F(x) \Delta x = \int_{x_i}^{x_f} F(x) dx$$

## Work Done by a Variable Force

In one dimension, the work done by a force  $F(x)$ , in moving an object from position  $x_i$  to position  $x_f$  is:

$$W = \int_{x_i}^{x_f} F(x) dx$$

Note that another way of describing this is to say that **the work done in moving an object from position  $x_i$  to position  $x_f$  is the area under the  $F(x)$  versus  $x$  curve, from  $x_i$  to  $x_f$ .**

Generalizing to 3 dimensions:

If a force acts to move an object from vector position  $\mathbf{r}_i$  to position  $\mathbf{r}_f$  the work done is:

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \vec{F}(\mathbf{r}) \cdot d\vec{r} = \int_{x_i}^{x_f} F dx + \int_{y_i}^{y_f} F dy + \int_{z_i}^{z_f} F dz$$

## The Work-Kinetic Energy Theorem and Variable Forces

We derived the work kinetic energy theorem for constant forces. In section 7-5 in the book it is shown that the work-kinetic energy theorem also applies for variable forces:

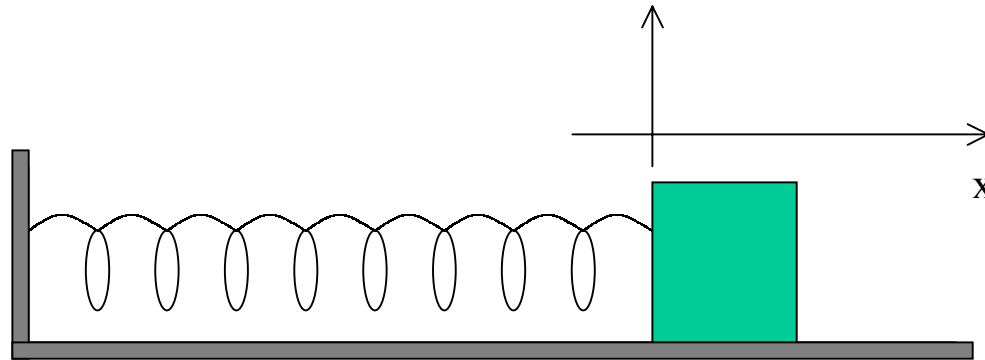
$$W = \Delta K = K_f - K_i$$

This theorem says that the work done on an object by a specific force is equal to the change in kinetic energy of the object.

However, just as for constant forces, **this theorem only applies if the only energy changed by the application of the force is the kinetic energy.**

## Work Done by A Spring Force

As an example of a force which depends on position, let's consider an ideal spring. In the picture below, a spring is attached to a wall. The other end of the spring is attached to a block with mass  $m$ . The spring is initially unstretched - and in this configuration the spring exerts no force on the block. We will mark position of the end of the unstretched spring as  $x=0$ .



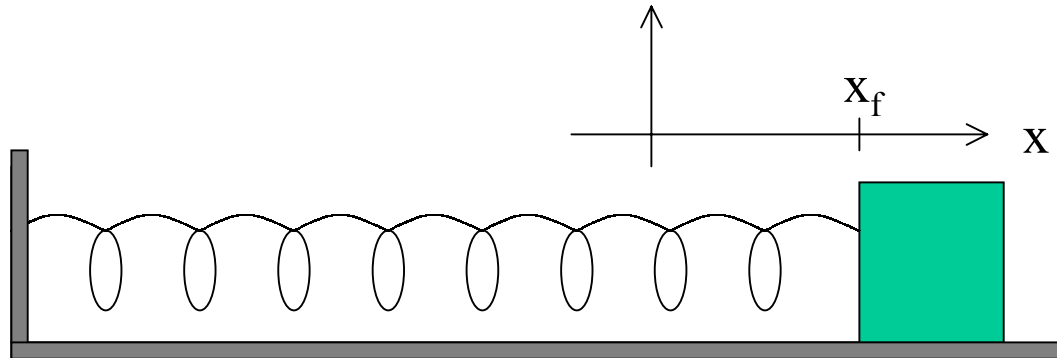
Spring is not stretched, force exerted by spring is zero.

## Work Done by A Spring Force

Now the spring is stretched a distance  $x_f$ , from its unstretched length. Experimentally we find that the spring exerts a force on the block:

$$F = -kx_f$$

where  $k$  is called the spring constant, and is dependent on the type of spring we are using.  $k$  has units of N/m.



## Work Done by a Spring Force

The above expression is called Hooke's Law, and can be written for any displacement  $x$  from the unstretched length of the spring:

$$F(x) = -kx$$

NOTE:

- 1) If  $x$  is positive (meaning the spring is stretched a distance  $x$ ), then the force exerted by the spring is negative (the spring will pull in the direction opposite to the displacement).
- 2) If  $x$  is negative (meaning the spring is compressed a distance  $x$ ), then the force exerted by the spring is positive (the spring will push in the direction opposite to the displacement).

## Work Done by a Spring Force

The work done by a spring as it moves from position  $x_i$  to position  $x_f$  (both measured from the unstretched length) **is not**  $W = F(x_f - x_i)$  .

Why? Because  $F$  varies with position. To determine the work done, **we need to integrate:**

$$W = \int_{x_i}^{x_f} F(x) dx$$

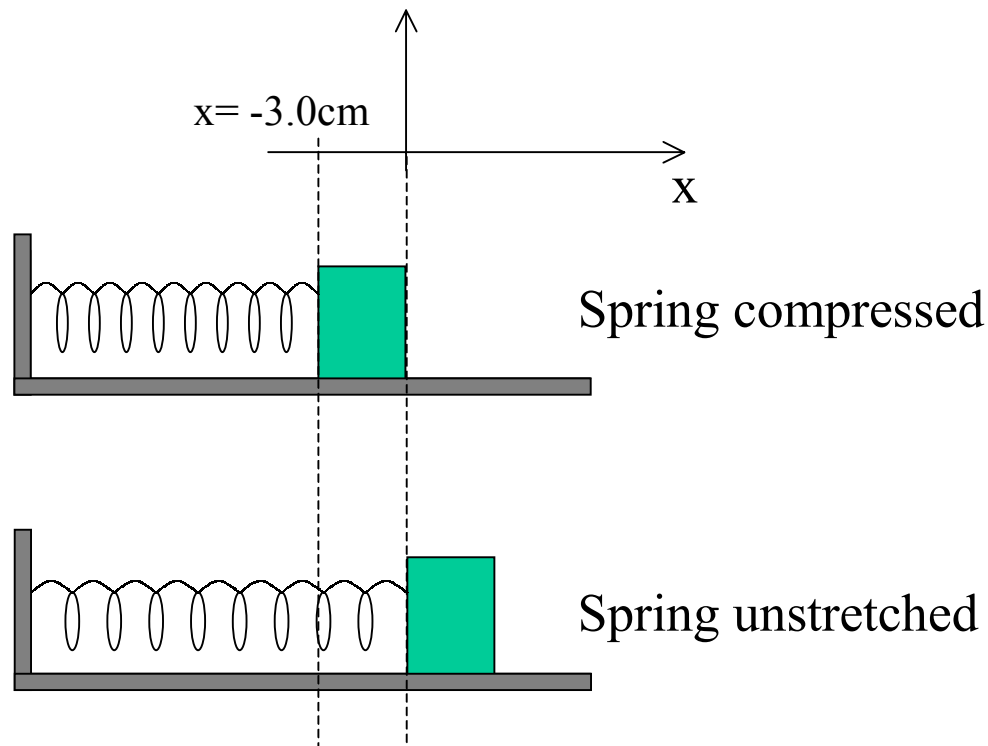
$$W = \int_{x_i}^{x_f} (-kx) dx = -k \int_{x_i}^{x_f} x dx = \left[ -\frac{1}{2} kx^2 \right]_{x_i}^{x_f}$$

$$W = -\frac{1}{2} k(x_f^2 - x_i^2)$$

## Example: Spring Force

A block is lying on a smooth surface, and is connected to a spring with spring constant  $80\text{N/m}$ . The spring is compressed a distance of  $3.0\text{cm}$  from its unstretched length, and then released.

- A) What is the work done by the spring force on the block as the block moves back towards the unstretched position?
- B) If the block was initially at rest, what is its speed when it reaches the unstretched position?
- C) Draw the force vs position curve for this problem.



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## Example (cont): Spring Force

A) Work done by the spring force:

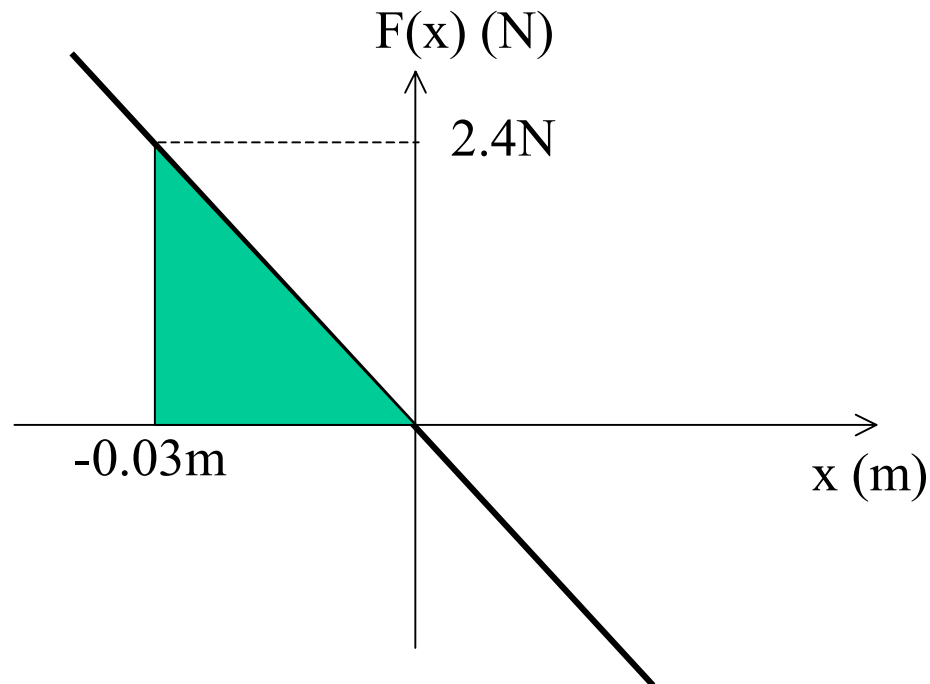
$$W = -\frac{1}{2}k(x_f^2 - x_i^2)$$
$$W = -\frac{1}{2}(80\text{N/m})[0^2 - (-0.03)^2]$$
$$W = +0.036\text{ J}$$

B) Since the only energy changed by spring force is the kinetic energy of the block, we can apply the work-energy theorem:

$$W = \Delta K = K_f - K_i$$
$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$
$$0.036\text{J} = \frac{1}{2}(10.0\text{kg})v_f^2 - 0$$
$$v_f = \sqrt{\frac{2(0.036)}{10.0}} = 0.085\text{m/s}$$

## Example (cont): Spring Force

C) The area under the force vs position curve from  $x = -0.03\text{m}$  to  $x = 0.0\text{m}$  is equal to the work done by the spring in moving the block between those 2 positions.



# Power

In many situations, it is important to know not just how much work is required, but how quickly that work needs to be done. The concept of Power does this.

Power is defined as the time rate of doing work. Another way of describing power is to say that power is the rate at which an applied force transfers energy to an object.

If a force does an amount of work  $\Delta W$  on an object in an amount of time  $\Delta t$ , then the average power due to the force is:

$$P = \frac{\Delta W}{\Delta t}$$

The units of power are Watts, written as W, where  $1\text{ W} = 1\text{ J/s}$ .

Don't confuse Watts (W) with Work (W).

## Power

The instantaneous power is the limiting value of the average power as  $\Delta t$  goes to zero:

$$P = \frac{dW}{dt}$$

Since the work done by a force in moving an object through a displacement  $dx$  is  $dW = F dx$ , we can also express power as:

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{x}}{dt} = \vec{F} \cdot \vec{v}$$

## Example: Power

A runner with mass 50.0kg runs up the stairs of the Sears Tower (which has a height of 443m). She accomplishes this in 15.0 minutes.

What was her average power output? Assume she runs at constant velocity.

Her average vertical component of her velocity is:

$$v_{y,\text{avg}} = \frac{443\text{m}}{900\text{s}} = 0.492\text{m/s}$$

The force she exerts is equal to her weight  $mg$ :

$$F = mg = (50\text{kg})(9.80\text{m/s}^2) = 490\text{N}$$

Her power output is then

$$P = Fv = (490\text{N})(0.492\text{m/s}) = 241\text{W}$$