Usual choice: common eigenbasis of $\hat{H}, \hat{L}^2, \hat{L}_z$.

12.5 The eigenvalue problem of $\hat{L}^2$ and $\hat{L}_z$

We will solve this eigenvalue problem algebraically, as we did (the second time around) with the harmonic oscillator.

For the harmonic oscillator solution we had to steps:

(1) Write $\hat{a}|0\rangle = 0$ in x-basis as $(\xi + \frac{\partial}{\partial \xi})\psi_0(\xi) = 0$

to get $\psi_0(\xi) \sim e^{-\xi^2/2}$

(2) Establish ladder algorithm to compute $|n\rangle$ from $|0\rangle$

as $|n\rangle = \frac{\hat{a}^{+n}}{\sqrt{n!}}|0\rangle$

For the angular momentum problem, we start with step (2).

Let's start by writing the common eigenvalue problem as

$$\hat{L}^2 |\alpha\beta\rangle = \alpha |\alpha\beta\rangle$$
$$\hat{L}_z |\alpha\beta\rangle = \beta |\alpha\beta\rangle$$

where α is eigenvalue of $\hat{L}^2$, β is eigenvalue of $\hat{L}$, and the pair of eigenvalues labels the common eigenstates. (These may still be degenerate - we will have to see about this later.)

Now, with 20/20 hindsight, I define raising and lowering operators:

$$\boxed{\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y}$$

They obviously satisfy $[\hat{L}^2, \hat{L}_\pm] = 0$, but also

$$\boxed{[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm} \qquad \left(\begin{aligned}[\hat{L}_z, \hat{L}_x \pm i\hat{L}_y] &= i\hbar L_y \pm i(-i\hbar L_x) \\ &= \pm\hbar(\hat{L}_x \pm i\hat{L}_y) = \pm\hat{L}_\pm \checkmark \end{aligned} \right)$$

This last relation is the key one, since

$$\hat{L}_z\left(\hat{L}_+|\alpha\beta\rangle\right) = (\hat{L}_+\hat{L}_z + \hbar\hat{L}_+)|\alpha\beta\rangle = (\beta+\hbar)\hat{L}_+|\alpha\beta\rangle$$

and
$$\hat{L}_z\left(\hat{L}_-|\alpha\beta\rangle\right) = (\hat{L}_-\hat{L}_z - \hbar\hat{L}_-)|\alpha\beta\rangle = (\beta-\hbar)\hat{L}_-|\alpha\beta\rangle$$

as well as
$$\hat{L}^2\hat{L}_\pm|\alpha\beta\rangle = \hat{L}_\pm\hat{L}^2|\alpha\beta\rangle = \alpha\hat{L}_\pm|\alpha\beta\rangle$$

So, if $|\alpha\beta\rangle$ is an eigenstate of $\hat{L}^2, \hat{L}_z$ with eigenvalues $\alpha\beta$,
then $\hat{L}_\pm|\alpha\beta\rangle$ " " " " " " " $\alpha, \beta\pm\hbar$.

$$\Rightarrow \quad \hat{L}_+|\alpha\beta\rangle = C_+(\alpha,\beta)|\alpha,\beta+\hbar\rangle$$
$$\hat{L}_-|\alpha\beta\rangle = C_-(\alpha,\beta)|\alpha,\beta-\hbar\rangle$$

So for every state $|\alpha\beta\rangle$ there exists an entire ladder of states $\ldots|\alpha,\beta-2\hbar\rangle, |\alpha,\beta-\hbar\rangle, |\alpha\beta\rangle, |\alpha,\beta+\hbar\rangle, |\alpha,\beta+2\hbar\rangle, \ldots$, seemingly infinite in both directions.

But this can't be true, since for finite $\hat{L}^2$, L_z can't be infinitely large! We expect classically $|L_z| \le \sqrt{L^2}$!

In quantum mechanics

$$\langle\alpha\beta|\hat{L}^2 - \hat{L}_z^2|\alpha\beta\rangle = \langle\alpha\beta|\hat{L}_x^2 + \hat{L}_y^2|\alpha\beta\rangle \ge 0 \Rightarrow \alpha - \beta^2 \ge 0$$
$$\Rightarrow \boxed{\beta^2 \le \alpha}$$

This can only mean that the ladder must break off, in both directions.

I.e. for given α, there must be a state $|\alpha, \beta_{max}\rangle$ such that $\boxed{\hat{L}_+ |\alpha, \beta_{max}\rangle = 0}$ (1)

and state $|\alpha, \beta_{min}\rangle$ such that $\boxed{\hat{L}_- |\alpha, \beta_{min}\rangle = 0}$. (2)

We can operate on (1) with $\hat{L}_+$ and use

$$\boxed{\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar \hat{L}_z}$$

$$\Rightarrow \quad 0 = \hat{L}_-\hat{L}_+|\alpha, \beta_{max}\rangle = (\alpha - \beta_{max}^2 - \hbar\beta_{max})|\alpha, \beta_{max}\rangle = 0$$

$$\Rightarrow \boxed{\alpha = \beta_{max}(\beta_{max} + \hbar)}$$

Similarly

$$0 = \hat{L}_+\hat{L}_-|\alpha, \beta_{min}\rangle = (\hat{L}^2 - \hat{L}_z^2 + \hbar \hat{L}_z)|\alpha, \beta_{min}\rangle = (\alpha - \beta_{min}^2 + \hbar\beta_{min}) \cdot |\alpha, \beta_{min}\rangle$$

$$\Rightarrow \boxed{\alpha = \beta_{min}(\beta_{min} - \hbar)}$$

Comparison $\Rightarrow \boxed{\beta_{min} = -\beta_{max}}$ (no surprise here!)

Between β_{min} and β_{max} there is an integer number of $\hbar$-steps:

$$\beta_{max} - \beta_{min} = 2\beta_{max} = k\hbar \Rightarrow \boxed{\beta_{max} = \frac{k\hbar}{2}} \quad k = 0, 1, 2 \ldots$$

$$\Rightarrow \boxed{\alpha = \beta_{max}(\beta_{max} + \hbar) = \hbar^2\left(\frac{k}{2}\right)\left(\frac{k}{2} + 1\right)}$$

We will refer to $\frac{k}{2} = \frac{\beta_{max}}{\hbar}$ as the <u>angular momentum</u> (in units of $\hbar$) of the state.

Interesting: unlike the classical case, where L_z can be as large as L (if $L_x = L_y = 0$), β_{max}^2 is smaller than α (except for the case when both $\alpha = \beta_{max} = 0$).

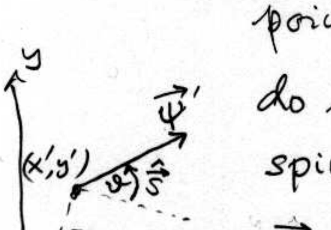
Note: For odd k, $\frac{L_z}{\hbar}$ has half-integer eigenvalues. But in Section 12.3, where we solved the eigenvalue problem for $\hat{L}_z$, we found only integer eigenvalues $m\hbar$. What's going on?

The answer to this puzzle is that, yes, $\hat{L}_z \leftrightarrow -i\hbar\frac{\partial}{\partial\varphi}$ has only integer eigenvalues. Here we did not use the explicit expressions for the $\hat{L}_i$, but only their commutator algebra. This commutator algebra allows for more solutions that are not realized for the orbital angular momentum operator $\hat{\vec{L}}$.

The more general solutions of the algebra $\hat{\vec{L}} \times \hat{\vec{L}} = i\hbar\hat{\vec{L}}$ with half-integer L_z-eigenvalues correspond to rotations of wavefunctions of more general form (vectors or spinors). For example if

$$\vec{\Psi}(x,y,z) = \psi_x(x,y,z)\vec{e}_x + \psi_y(x,y,z)\vec{e}_y + \psi_z(x,y,z)\vec{e}_z$$

its rotated version is to (i) rotate the vector and (ii) assign the rotated vector to a rotated point (x',y',z'). $\hat{\vec{L}}$ does only part (ii); to also do part (i) we need additional generators called spin $\hat{S}_x, \hat{S}_y, \hat{S}_z$ which shuffle the components ψ_x, ψ_y, ψ_z of $\vec{\Psi}$:



$$\rightarrow \boxed{\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}} \quad \text{with} \quad \hat{\vec{J}} \times \hat{\vec{J}} = i\hbar\hat{\vec{J}}}$$

The spin angular momentum $\hat{S}$ can have half-integer eigenvalues for $\hat{S}_z$.

Our analysis of the angular momentum algebra, together with the earlier observation that $\hat{L}_z$ has only integer eigenvalues, tells us (even though we know practically nothing about spin so far) that integer and half-integer eigenvalues for $\hat{S}_z$ are the only possible values.

So this is what have found so far:

- The eigenvectors $|jm\rangle$ of $\hat{J}^2, \hat{J}_z$ satisfy

$$\hat{J}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle \quad \text{with eigenvalues } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$\hat{J}_z |jm\rangle = m\hbar |jm\rangle \quad \text{with eigenvalues } m = \underbrace{j, j-1, j-2, \ldots, -j}_{(2j+1) \text{ states}}$$

We call j the angular momentum (in units of $\hbar$) and m the magnetic quantum number.

- For $\hat{\vec{J}} = \hat{\vec{L}} = \hat{\vec{X}} \times \hat{\vec{P}}$ (only orbital angular momentum) we have

$$\hat{L}^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle \qquad l = 0, 1, 2, \ldots$$

$$\hat{L}_z |lm\rangle = m\hbar |lm\rangle \qquad m = \underbrace{l, l-1, l-2, \ldots, -l}_{(2l+1) \text{ states for each } l.}$$

So we found the spectrum of eigenvalues. What about the associated eigenfunctions?

N.B. The ladders connecting $|l,-l\rangle$ with $|ll\rangle$ via $\hat{L}_+$ are unique $\rightarrow$ no degeneracy of m!

We found that

$$\hat{J}_\pm |jm\rangle = C_\pm(j,m)\,|j,m\pm 1\rangle$$

Let's now find the $C_\pm(j,m)$

$$\left(\hat{J}_+|jm\rangle = C_+(j,m)\,|j,m+1\rangle\right)^\dagger = \langle jm|\hat{J}_- = C_+^*(j,m)\,\langle j,m+1|$$

$$\Rightarrow \underbrace{\langle jm|\hat{J}_-\hat{J}_+|jm\rangle}_{} = |C_+(j,m)|^2\langle j,m+1|j,m+1\rangle = |C_+(j,m)|^2$$

$$= \langle jm|\hat{J}^2-\hat{J}_z^2-\hbar J_z|jm\rangle = \left(j(j+1)\hbar^2-m^2\hbar^2-m\hbar^2\right)\underbrace{\langle jm|jm\rangle}_{1}$$

$$\Rightarrow C_+(j,m) = \hbar\sqrt{(j-m)(j+m+1)}\,.$$ (choosing zero phase)

Similarly one finds

$$C_-(j,m) = \hbar\sqrt{(j+m)(j-m+1)}$$

$$\Rightarrow \boxed{\hat{J}_\pm|jm\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\;|j,m\pm 1\rangle}$$

use this to construct whole ladder starting from $|jj\rangle$ or $|j,-j\rangle$.

When $\hat{J}_\pm$ acts on $|j,\pm j\rangle$ they will kill the state ✓

From these results we get the matrix elements for $\hat{J}_x$ and $\hat{J}_y$ as follows:

$$\langle j'm'|\hat{J}_x|jm\rangle = \langle j'm'|\,\frac{\hat{J}_++\hat{J}_-}{2}\,|jm\rangle = \frac{\hbar}{2}\delta_{jj'}\cdot\left\{\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} + \sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\right\}$$

$$\langle j'm'|\underbrace{\hat{J}_y}_{\frac{1}{2i}(\hat{J}_+-\hat{J}_-)}|jm\rangle = \frac{\hbar}{2i}\delta_{jj'}\left\{\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} - \sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\right\}$$

So in the $|jm\rangle$ basis $\hat{J}^2, \hat{J}_z$ are diagonal while $\hat{J}_x, \hat{J}_y$ are off diagonal (they mix states with different m, m') but block diagonal (they don't mix blocks with different j).

See p. 328 for explicit form of the matrices

$$\langle j'm' | \hat{J}^2, \hat{J}_x, \hat{J}_y | mj \rangle .$$

Since the $\hat{J}_i$ are blockdiagonal, so is $\vec{\theta} \cdot \vec{\hat{J}}$ and the rotation matrix $\hat{U}[R(\vec{\vartheta})] = e^{-\frac{i}{\hbar}\vec{\vartheta}\cdot\vec{\hat{J}}}$:

Each j corresponds to a $(2j+1) \times (2j+1)$ dimensional matrix that operates on $(2j+1)$-d vectors $\begin{pmatrix} |jj\rangle \\ |j, j-1\rangle \\ \vdots \\ |j, -j\rangle \end{pmatrix}$.

$\hat{U}$ doesn't mix components of $|\psi\rangle$ in the subspace $\mathbb{V}_j$ spanned by $\{|j,j\rangle, \ldots, |j,-j\rangle\}$ with any states from other subspaces $\mathbb{V}_{j' \neq j}$. So we need only to consider

$$\hat{D}^{(j)}[R(\vec{\vartheta})] = e^{-\frac{i}{\hbar}\vec{\vartheta}\cdot\vec{\hat{J}}^{(j)}}$$

where in the $|jm\rangle$ basis $\hat{J}^{(j)}$ is represented by a $(2j+1) \times (2j+1)$ dimensional matrix, and $\hat{U}[R(\vec{\vartheta})]$ is represented by

$$\hat{U}[R] \xrightarrow[\text{basis}]{|jm\rangle} \begin{pmatrix} \boxed{D^{(0)}} & & \\ & \boxed{D^{(1/2)}} & \\ & & \underbrace{\boxed{D^{(1)}}}_{2j+1} \Big\} 2j+1 \end{pmatrix}$$

$$\begin{aligned}\langle j'm'|J_y|jm\rangle &= \langle j'm'|\,\frac{J_+-J_-}{2i}\,|jm\rangle \\ &= \frac{\hbar}{2i}\{\delta_{jj'}\,\delta_{m',m+1}[(j-m)(j+m+1)]^{1/2}-\delta_{jj'}\,\delta_{m',m-1} \\ &\qquad \times[(j+m)(j-m+1)]^{1/2}\}\end{aligned} \tag{12.5.21b}$$

Using these (or our mnemonic based on images) we can write down the matrices corresponding to J^2, J_z, J_x, and J_y in the $|jm\rangle$ basis‡:

$$J^2 \rightarrow \begin{array}{c|c} {}_{j'm'}\diagdown^{jm} & (0,0)\quad (\tfrac{1}{2},\tfrac{1}{2})\quad (\tfrac{1}{2},-\tfrac{1}{2})\quad (1,1)\quad (1,0)\quad (1,-1)\quad \cdots \\ \begin{matrix}(0,0)\\(\frac{1}{2},\frac{1}{2})\\(\frac{1}{2},-\frac{1}{2})\\(1,1)\\(1,0)\\(1,-1)\\ \vdots\end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{3}{4}\hbar^2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & \frac{3}{4}\hbar^2 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 2\hbar^2 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 2\hbar^2 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 2\hbar^2 & \\ \vdots & & & & & & \ddots \end{bmatrix}\end{array} \tag{12.5.22}$$

J_z is also diagonal with elements $m\hbar$.

$$J_x \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \hbar/2 & 0 & 0 & 0 & \\ 0 & \hbar/2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \hbar/2^{1/2} & 0 & \\ 0 & 0 & 0 & \hbar/2^{1/2} & 0 & \hbar/2^{1/2} & \\ 0 & 0 & 0 & 0 & \hbar/2^{1/2} & 0 & \\ \vdots & & & & & & \ddots \end{bmatrix} \tag{12.5.23}$$

$$J_y \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -i\hbar/2 & 0 & 0 & 0 & \\ 0 & i\hbar/2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -i\hbar/2^{1/2} & 0 & \\ 0 & 0 & 0 & i\hbar/2^{1/2} & 0 & -i\hbar/2^{1/2} & \\ 0 & 0 & 0 & 0 & i\hbar/2^{1/2} & 0 & \\ \vdots & & & & & & \ddots \end{bmatrix} \tag{12.5.24}$$

Notice that although J_x and J_y are not diagonal in the $|jm\rangle$ basis, they are *block diagonal*: they have no matrix elements between one value of j and another. This is

‡ The quantum numbers j and m do not fully label a state; a state is labeled by $|\alpha jm\rangle$, where α represents the remaining labels. In what follows, we suppress α but assume it is the same throughout.

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Exercise 12.5.2. (1)
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(2) Do the same fo
(3) Construct the 4

*Exercise 12.5.3.** (1
(2) Show that in th

(use symmetry argumen
(3) Check that ΔJ_x
principle [Eq. (9.2.9)].
(4) Show that the

Finite Rotations‡

Now that we hav
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element $|\psi_j'\rangle$ of $\mathbb{V}_j$. Th
if $|\psi\rangle$ has componen
matrices $D^{(j)}$. What m
evaluate these if j is sm

$D^{(j)}[R$

‡ The material from here to

For spherically symmetric systems, $[\hat{H}, \hat{J}^2] = [\hat{H}, \hat{J}_z] = 0$,

hence $\hat{H}$ is diagonal in the $|jm\rangle$ basis and

within each j-block, $\hat{H}$ has the same eigenvalue

E_j (since $[\hat{H}, \hat{J}_\pm] = 0$).

$\Rightarrow$ all states in $\mathbb{V}_j$ are energetically degenerate for rotationally invariant Hamiltonians

Angular momentum eigenfunctions in the coordinate basis

If we write $\hat{\vec{X}}$ and $\hat{\vec{P}}$ in the x-basis and work out from them the x-basis matrix elements of $\hat{\vec{L}} = \hat{\vec{X}} \times \hat{\vec{P}}$ we find

$$\hat{L}_z \leftrightarrow -i\hbar \frac{\partial}{\partial\varphi}$$

$$\hat{L}_\pm \leftrightarrow \pm\hbar e^{\pm i\varphi} \left(\frac{\partial}{\partial\vartheta} \pm i \cot\vartheta \frac{\partial}{\partial\varphi} \right)$$

$$\hat{L}_x \leftrightarrow i\hbar \left(\sin\varphi \frac{\partial}{\partial\vartheta} + \cos\varphi \cot\vartheta \frac{\partial}{\partial\varphi} \right)$$

$$\hat{L}_y \leftrightarrow i\hbar \left(-\cos\varphi \frac{\partial}{\partial\vartheta} + \sin\varphi \cot\vartheta \frac{\partial}{\partial\varphi} \right)$$

Start from $\hat{L}_+ |\ell\ell\rangle = 0$

$$\Rightarrow \left(\frac{\partial}{\partial\vartheta} + i\cot\vartheta \frac{\partial}{\partial\varphi} \right) Y_\ell^\ell(\vartheta,\varphi) = 0$$

Since $Y_\ell^\ell(\vartheta,\varphi)$ is an eigenfunction of $\hat{L}_z \leftrightarrow -i\hbar\frac{\partial}{\partial\varphi}$ with eigenvalue $\ell\hbar$, we can write

$$Y_\ell^\ell(\vartheta,\varphi) = \mathcal{U}_\ell^\ell(\vartheta)e^{i\ell\varphi}$$

where $\left(\frac{\partial}{\partial\vartheta} - \ell\cot\vartheta\right)\mathcal{U}_\ell^\ell(\vartheta) = 0$

or $\frac{d\mathcal{U}}{\mathcal{U}} = \ell\,\frac{d(\sin\vartheta)}{\sin\vartheta}$ $\qquad (\cos\vartheta\, d\vartheta = d(\sin\vartheta))$

$\Rightarrow \underline{\mathcal{U}_\ell^\ell(\vartheta) = R(\sin\vartheta)^\ell}$ $\qquad$ (R contains the radial dependence of the total wave function $\psi(r,\vartheta,\varphi)$)

If we normalize $Y_\ell^\ell(\vartheta,\varphi)$ over (ϑ,φ),

$$\boxed{\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\vartheta)\left|Y_\ell^\ell(\vartheta,\varphi)\right|^2 = 1}$$

We get $\boxed{Y_\ell^\ell(\vartheta,\varphi) = (-1)^\ell \sqrt{\frac{(2\ell+1)!}{4\pi}}\,\frac{1}{2^\ell \ell!}\sin^\ell\vartheta\, e^{i\ell\varphi}}$

convention

We can now get $Y_\ell^{\ell-1}(\vartheta,\varphi)$ by applying $\hat{L}_-$:

$$\hat{L}_-|\ell\ell\rangle = \hbar\sqrt{(\ell+\ell)(\ell-\ell+1)}\,|\ell,\ell-1\rangle = \sqrt{2\ell}\,\hbar\,|\ell,\ell-1\rangle$$

$$\Rightarrow Y_\ell^{\ell-1}(\vartheta,\varphi) = \frac{1}{\sqrt{2\ell}}\left(\frac{-1}{\hbar}\right)\left[\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\vartheta} - i\cot\vartheta\,\frac{\partial}{\partial\varphi}\right)\right]Y_\ell^\ell(\vartheta,\varphi)$$

and so on. One finds for $m \geq 0$

$$Y_\ell^m(\vartheta,\varphi) = (-1)^\ell \sqrt{\frac{(2\ell+1)!}{4\pi}}\,\frac{1}{2^\ell \ell!}\sqrt{\frac{(\ell+m)!}{(2\ell)!\,(\ell-m)!}}\; e^{im\varphi}\,\frac{1}{\sin^m\vartheta}\,\frac{d^{\ell-m}}{d(\cos\vartheta)^{\ell-m}}\sin^{2\ell}\vartheta$$

$$Y_\ell^{-m}(\vartheta,\varphi) = (-1)^m\, Y_{\ell m}^*(\vartheta,\varphi)$$

These functions are called <u>spherical harmonics</u>

and satisfy

$$\underbrace{\int d\Omega}_{=\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\vartheta)} \left(Y_\ell^m(\vartheta,\varphi)\right)^* Y_{\ell' m'}(\vartheta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'} = \langle \ell m | \ell' m' \rangle .$$

They are also written as

$$\boxed{Y_{\ell m}(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\vartheta)\, e^{im\varphi}}$$

where $P_\ell^m(x)$ are the <u>associated Legendre polynomials</u>.

$P_\ell(x) \equiv P_\ell^0(x)$ are called <u>Legendre polynomials</u>.

The first few spherical harmonics are

$\ell=0$: $\quad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$\ell=1$: $\quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\vartheta \;,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{\pm i\varphi}$

$\ell=2$: $\quad Y_{20} = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\vartheta - 1)\,,\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\vartheta\cos\vartheta\, e^{\pm i\varphi}$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\vartheta\, e^{\pm 2i\varphi}$$

For large ℓ, $Y_{\ell\ell} \sim \sin^\ell\vartheta$ is almost entirely confined to x-y plane, as classically expected for a particle with $\vec{L} \parallel \vec{e}_z$.

For large ℓ, $Y_{\ell 0}$ is almost entirely confined to $\vartheta \simeq 0, \pi$, i.e. to the z axis.