Example Scattering off a hard sphere

$$V(r) = \begin{cases} \infty & r < r_0 \\ 0 & r > r_0 \end{cases}$$

1) Solve S. Eq.
2) Look at soln. for $r \to \infty$ and identify phase shifts.

$$R_\ell(r) = \begin{cases} 0 & r < r_0 \\ A_\ell j_\ell(kr) + B_\ell n_\ell(kr) & \end{cases}$$

Boundary condition at r_0:

$$R_\ell(r_0) = 0 = A_\ell j_\ell(kr_0) + B_\ell n_\ell(kr_0)$$

$$\Rightarrow \quad \frac{B_\ell}{A_\ell} = -\frac{j_\ell(kr_0)}{n_\ell(kr_0)} \equiv -\tan\delta_\ell \qquad \Rightarrow \begin{array}{l} A_\ell = N\cos\delta_\ell \\ B_\ell = -N\sin\delta_\ell \\ N = +\sqrt{A_\ell^2 + B_\ell^2} \end{array}$$

Asymptotically

$$R_\ell(r) \xrightarrow[r\to\infty]{} \frac{1}{kr}\left[A_\ell \sin\left(kr - \frac{\ell\pi}{2}\right) - B_\ell \cos\left(kr - \frac{\ell\pi}{2}\right)\right]$$

$$= \frac{N}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)$$

So we have

$$\boxed{\delta_\ell = \tan^{-1}\left(\frac{j_\ell(kr_0)}{n_\ell(kr_0)}\right)} \quad \text{(hard sphere)}$$

For s-waves this gives

$$\delta_0 = \tan^{-1}\left(\frac{\sin(kr_0)/kr_0}{-\cos(kr_0)/kr_0}\right) = \tan^{-1}\left(-\tan(kr_0)\right) = -kr_0$$

The hard sphere has pushed out the wave function, which starts to oscillate at $r=r_0$ instead of $r=0$

$\rightarrow$ phase shift $-kr_0$.

In general: repulsive potentials give negative phase shifts (they slow down the particle, reducing the WKB phase)

attractive potentials give positive phase shifts

(true as long as $2\delta_\ell < \pi$)

For $kr_0 = \pi$, $a_0 = 0$ $\Rightarrow$ s-waves don't "see" the hard sphere at that energy!

- What is the low-energy limit of the hard-sphere phase shifts?

use
$$\left.\begin{aligned} j_\ell(x) &\xrightarrow[x\to 0]{} \frac{x^\ell}{(2\ell+1)!!} \\ n_\ell(x) &\xrightarrow[x\to 0]{} -\frac{1}{(2\ell-1)!!}\frac{1}{x^{\ell+1}} \end{aligned}\right\} \frac{j_\ell}{n_\ell} \rightarrow \sim x^{2\ell+1} \xrightarrow[x\to 0]{} 0$$

$$\Rightarrow \boxed{\tan\delta_\ell \xrightarrow[k\to 0]{} \delta_\ell = -\frac{(kr_0)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}}$$

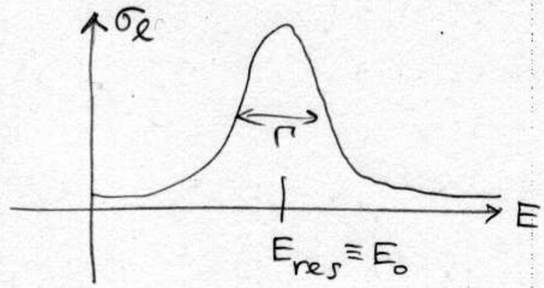
Scattering in high-ℓ states is suppressed; $\ell=0$ (isotropic scattering) dominates.

Note: $\boxed{\sigma_{\ell=0} \sim \frac{\sin^2\delta_0}{k^2} \xrightarrow[k\to 0]{} \text{const} \neq 0}$ as $k\to 0$,

so the scattering cross section remains finite in the zero-energy limit!

Scattering resonances

Even though at low energies the partial cross section σ_ℓ is generically small, $\delta_\ell \sim k^{2\ell+1}$, sometimes the phase shift δ_ℓ happens to rise very quickly from 0 to π (or from $n\pi$ to $(n+1)\pi$) over a narrow range of k (or E).
When this happens, the cross section develops a characteristic peak structure as function of energy, called a scattering resonance:



If δ_ℓ rises rapidly near $E_0 = \dfrac{\hbar^2 k_0^2}{2\mu}$, then

we can write

$$\delta_\ell = \delta_b + \tan^{-1}\left(\frac{\Gamma/2}{E_0 - E}\right)$$

"background phase", $\approx$ energy independent over the region Γ

(at $E = E_0$, $\delta_\ell = \frac{\pi}{2}$;
at $E \ll E_0$, $E_0 \gg \Gamma$, $\delta_\ell \approx n\pi$;
at $E \gg E_0 + \Gamma$, $\delta_\ell \approx (n+1)\pi$)

Neglecting δ_b, we get

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell \underset{|E-E_0|\lesssim\Gamma}{\approx} \frac{4\pi}{k^2}(2\ell+1)\sin^2\left(\operatorname{atan}\left(\frac{\Gamma/2}{E_0 - E}\right)\right)$$

$$\operatorname{arctg}\frac{1}{x} = \operatorname{arc\,ctg}(x) = \arcsin\frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \sin\left(\operatorname{arctg}\left(\frac{\Gamma/2}{E_0-E}\right)\right) = \sin\left(\arcsin\frac{1}{\sqrt{1+\left(\frac{E_0-E}{\Gamma/2}\right)^2}}\right)$$

$$= \frac{\Gamma/2}{\sqrt{(E_0-E)^2+\Gamma^2/4}}$$

$$\Rightarrow \boxed{\sigma_\ell \underset{|E-E_0|\lesssim\Gamma}{\approx} \frac{4\pi}{k^2}(2\ell+1)\frac{\Gamma^2/4}{(E_0-E)^2+\Gamma^2/4}}$$

Breit-Wigner formula

The maximum cross section at the resonance energy is

$$\sigma_\ell^{max} = \frac{4\pi}{k^2}(2\ell+1),$$ i.e., it exhausts the unitarity bound.

In deriving this we assumed that, near E_0, Γ can be treated as a constant. But Γ should really be k-dependent. The k-dependence can be deduced by noting that, for $k\to 0$,

$$\sigma_\ell \sim \frac{1}{k^2}\sin^2\delta_\ell \approx \frac{1}{k^2}\delta_\ell^2 \sim \frac{(kr_0)^{4\ell+2}}{k^2}$$

$$\Rightarrow \Gamma/2 = (kr_0)^{2\ell+1}\gamma$$. where γ has units of energy and is constant.

So a better approximation that is valid over a wider region in energy is

$$\boxed{\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{\gamma^2(kr_0)^{4\ell+2}}{(E-E_0)^2+\gamma^2(kr_0)^{4\ell+2}}}$$

This dampens, for $l \neq 0$, σ_l at low energies by a factor k^{4l} (only s-wave scattering survives), except at the resonance energies E_0 where it is compensated by a similar factor in the denominator.
As l goes up, the resonances get sharper (if you hold γ fixed).

What exactly is going on at a resonance? How is this related to the properties of the solutions of the Schrödinger equation? Let's look at the S-matrix:

Near a resonance we have

$$\underset{\sim}{S_l(k)} = e^{2i\delta_l(k)} = \frac{e^{i\delta_l}}{e^{-i\delta_l}} = \frac{\cos\delta_l + i\sin\delta_l}{\cos\delta_l - i\sin\delta_l}$$

$$= \frac{1+i\tan\delta_l}{1-i\tan\delta_l} = \frac{1+i\frac{\Gamma/2}{E_0-E}}{1-i\frac{\Gamma/2}{E_0-E}} = \underset{\sim}{\frac{E-E_0-i\Gamma/2}{E-E_0+i\Gamma/2}}$$

Let us think of this S_l as a function of complex k.
This function obviously has a pole at

$$\boxed{E = E_0 - i\frac{\Gamma}{2}} \quad \text{or} \quad \boxed{k = k_0 - i\frac{\eta}{2}}$$

where $\frac{\hbar^2 k_0^2}{2\mu} = E_0$ and $\eta = \frac{\mu\Gamma}{\hbar^2 k_0}$ (assuming that Γ and η are small enough that we can drop terms $\sim \Gamma^2$).

Since Γ and η are small compared with E_0 and k_0, respectively, the pole is close to the real axis.

So the peak in S_ℓ on the real axis arises from a pole nearby in the complex plane; the closer the pole is to the real axis ($\eta \to 0$), the narrower (sharper) the resonance becomes.

What is the meaning of a pole of the S-matrix in the complex plane?

Consider a bound state of $\hat{H}$ (assuming a finite range potential). To be normalizable, its wave function must fall off at large r exponentially:

$$R_{\kappa\ell}(r) \xrightarrow[r\to\infty]{} \frac{A e^{-\kappa r}}{r}$$

For generic κ values the solution reads

$$R_{\kappa\ell}(r) \xrightarrow[r\to\infty]{} \frac{A e^{-\kappa r}}{r} + \frac{B e^{+\kappa r}}{r}$$

but this is not normalizable; only for specific discrete κ values, corresponding to the energy eigenvalues, does the B-term disappear.

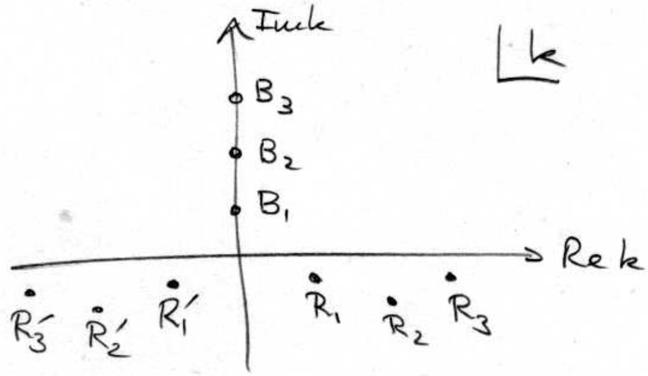
Compare this to scattering states at $E>0$, which behave asymptotically as

$$R_{k\ell}(r) \xrightarrow[r\to\infty]{} \frac{A e^{ikr}}{r} + \frac{B e^{-ikr}}{r}$$

Their phase shift and S-matrix factors are determined by (see discussion of p. 120)

$$e^{2i\delta_\ell(k)} = S_\ell(k) = \frac{A}{B} = \frac{\text{outgoing wave amplitude}}{\text{incoming wave amplitude}}.$$

Using this definition for the bound state (which has $B=0$) we see that a bound state corresponds to a pole of the S-matrix at $k = i\varkappa$, i.e. for purely imaginary k:



So resonances are also some weird kind of bound state, in the sense that they are complex poles of the S-matrix. The real bound state wave functions have time dependence

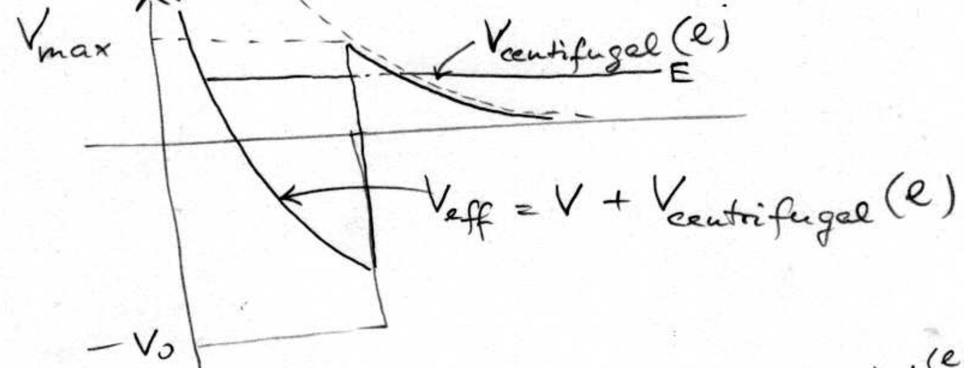
$$e^{-iE_B t/\hbar} \qquad \left(E_B = -\frac{\hbar^2 \varkappa_B^2}{2\mu} < 0 \right);$$

Correspondingly, the resonance at $E = E_0 - i\Gamma/2$ corresponds to a wave function with time dependence

$$e^{-i\left(E_0 - \frac{i\Gamma}{2}\right)t/\hbar} = e^{-iE_0 t/\hbar}\, e^{-\Gamma t/2\hbar} \quad \left(E_0 = \frac{\hbar^2 k_0^2}{2\mu} > 0\right).$$

This describes a state with positive energy $E_0 > 0$ whose norm decays exponentially with half-life $t \sim \frac{\hbar}{\Gamma}$. $\Rightarrow$ A pole at $E = E_0 - i\Gamma/2$ describes an unstable "bound" state of mean energy $E_0 > 0$ and lifetime $\tau = \frac{\hbar}{\Gamma}$. (Due to the finite lifetime, the energy of the state is not sharply defined, as seen by the width of the peak in the cross section $\sigma_\ell(E)$.)

How can a positive energy particle form a metastable bound state?! Consider a square well:



At nonzero ℓ, the effective potential $V_{eff}^{(\ell)}(r) = V(r) + \frac{\ell(\ell+1)}{2\mu r^2}$ develops a "pocket" that can sustain unstable quasibound states at $E > 0$ whose decay is inhibited by a potential barrier. The lifetime is controlled by the tunneling rate through the barrier. As ℓ increases, $V_{centrifugal}$ increases and tunneling is suppressed $\rightarrow$ the resonances get sharper. For $\ell = 0$ their are no resonances in this potential.