Finite translations

What is the operator for a finite translation a?

Divide a into N equal parts a/N, $N \to \infty$.

As $N \to \infty$, $T(a/N) = \mathbb{1} - \frac{ia}{\hbar N}\hat{P} + \mathcal{O}((\frac{a}{N})^2)$

$$\Rightarrow \underbrace{\hat{T}(a)} = \lim_{N\to\infty} \prod_{i=1}^{N} \hat{T}(a/N) = \lim_{N\to\infty} \left[\hat{T}(\tfrac{a}{N})\right]^N$$

$$= \lim_{N\to\infty} \left(\mathbb{1} - \frac{ia}{\hbar}\frac{\hat{P}}{N}\right)^N = \underbrace{e^{-ia\hat{P}/\hbar}}$$

In the x-basis this acts like

$$\hat{T}(a) \xrightarrow[x \text{ basis}]{} e^{-a\frac{d}{dx}}$$

$$\Longleftrightarrow \langle x|T(a)|\psi\rangle = \underbrace{\psi_a(x)} = e^{-a\frac{d}{dx}}\psi(x)$$

$$= \psi(x) - a\frac{d\psi}{dx} + \frac{a^2}{2!}\frac{d^2\psi}{dx^2} - + \dots$$

$$= \underbrace{\psi(x-a)} \quad \text{(Taylor expansion!)} \checkmark$$

Consistency check:

$$\hat{T}(a)\,\hat{T}(b) = e^{-ia\hat{P}/\hbar}\, e^{-ib\hat{P}/\hbar} = e^{-i(a+b)\hat{P}/\hbar} = \hat{T}(a+b) \checkmark$$

Note also that, since $[\hat{T}(a), \hat{H}] = 0$, the same holds for the time evolution operator $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$:

$$\boxed{[\hat{T}(a), \hat{U}(t)] = 0} \Rightarrow \text{translation invariance is preserved in time.}$$

System of N particles

$$\langle x_1, \dots x_N | \hat{T}(\varepsilon) | \psi \rangle = \psi(x_1 - \varepsilon, x_2 - \varepsilon, \dots, x_N - \varepsilon)$$

$$= \langle x_1, \dots, x_N | \hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \hat{P} | \psi \rangle + O(\varepsilon^2) = \psi(x_1 \dots x_N) - \sum_{i=1}^{N} \varepsilon \frac{\partial \psi}{\partial x_i}$$

$$\Rightarrow \boxed{\hat{T}(\varepsilon) = \hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \sum_{i=1}^{N} \hat{P}_i = \hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \hat{P}}$$

total momentum operator

One shows that

$$\left. \begin{aligned} \hat{T}(\varepsilon) \hat{X}_i \hat{T}(\varepsilon) &= \hat{X}_i + \varepsilon \hat{\mathbb{1}} \\ \hat{T}(\varepsilon) \hat{P}_i \hat{T}(\varepsilon) &= \hat{P}_i \end{aligned} \right\} i = 1, \dots, N$$

Translational invariance:

$$\hat{H}(\vec{\hat{X}}, \vec{\hat{P}}) = \hat{H}(\vec{\hat{X}} + \vec{\varepsilon} \hat{\mathbb{1}}, \vec{\hat{P}})$$

where $\vec{\hat{X}} = (\hat{X}_1, \dots, \hat{X}_N)$ etc.

and $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)$

Finite translations:

$$\hat{T}(\vec{a}) = e^{-i \vec{a} \cdot \vec{\hat{P}} / \hbar} \qquad \text{where } \vec{a} = (a_1, a_2, \dots, a_N)$$
$$\vec{\hat{P}} = (\hat{P}_1, \hat{P}_2, \dots, \hat{P}_N)$$

Works also for a single particle in N dimensions.

10.2 Time translation invariance

Consider infinitesimal time evolution of a state $|\psi_0\rangle$, prepared at time t_1:

$$|\psi(t_1+\varepsilon)\rangle = \left(\hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar}\hat{H}(t_1)\right)|\psi_0\rangle + O(\varepsilon^2)$$

If we prepare the same state and let it again evolve by a time ε:

$$|\psi(t_2+\varepsilon)\rangle = \left(\hat{\mathbb{1}} - \frac{i\varepsilon}{\hbar}\hat{H}(t_2)\right)|\psi_0\rangle + O(\varepsilon^2)$$

The evolved states will be identical if

$$0 = |\psi(t_2+\varepsilon)\rangle - |\psi(t_1+\varepsilon)\rangle = -\frac{i\varepsilon}{\hbar}\left(\hat{H}(t_2) - \hat{H}(t_1)\right)|\psi_0\rangle + O(\varepsilon^2)$$

If we require this to hold for all states $|\psi_0\rangle$ (time translation invariance of the physics), then

$$\hat{H}(t_2) = \hat{H}(t_1)$$

Since t_1, t_2 were arbitrary, it follows that $\hat{H}$ must be time-independent,

$$\boxed{\frac{\partial \hat{H}}{\partial t} = 0}$$

if time translation invariance is to hold.

$$\Rightarrow \boxed{\frac{d}{dt}\langle \hat{H}\rangle = \frac{dE}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{H}]\rangle = 0}$$ (from Ehrenfest with $\frac{\partial \hat{H}}{\partial t} = 0$)

energy conservation

11.3 Parity invariance

Parity is a <u>discrete</u> symmetry, unlike space-time translations which are continuous:

$$x \xrightarrow{\text{parity}} -x$$

$$p \longrightarrow -p$$

In QM, we describe this operation by the <u>parity operator</u> $\hat{\pi}$:

$$\hat{\pi}\,|x\rangle = |-x\rangle \quad (\neq -|x\rangle!)$$

$$\hat{\pi}\,|p\rangle = |-p\rangle$$

What consequences do these rules have for a general state $|\psi\rangle$?

$$\hat{\pi}\,|\psi\rangle = \hat{\pi}\left(\underbrace{\int_{-\infty}^{\infty} dx\,|x\rangle\langle x|}_{\hat{\mathbb{1}}}\psi\rangle\right) = \int_{-\infty}^{\infty} dx\,|-x\rangle\langle x|\psi\rangle$$

$$\overset{x'=-x}{=} \int_{-\infty}^{\infty} dx'\,|x'\rangle\langle -x'|\psi\rangle = \int_{-\infty}^{\infty} dx'\,\psi(-x')\,|x'\rangle$$

$$\Rightarrow \boxed{\langle x|\hat{\pi}|\psi\rangle = \int dx'\,\psi(-x')\,\underbrace{\langle x|x'\rangle}_{\delta(x-x')} = \psi(-x)}$$

So the wave function corresponding to the state $\hat{\pi}|\psi\rangle$ is $\psi(x)$ if the original state had wave function $\psi(x)$.

The eigenvalues of $\hat{\pi}$ are ± 1 since

$$\hat{\pi}^2|x\rangle = \hat{\pi}\left(\hat{\pi}|x\rangle\right) = \hat{\pi}|-x\rangle = |x\rangle \qquad (*)$$

So $\hat{\pi}^2$ has the same eigenstates as $\hat{\pi}$, and its eigenvalues are the squares of the eigenvalues of $\hat{\pi}$, hence the eigenvalues of $\hat{\pi}$ are $\sqrt{1} = \pm 1$.

Since $(*)$ holds for all* basis states, it holds for all states:

$$\hat{\pi}^2 = \hat{\mathbb{1}}$$

Further: $\left.\begin{array}{l} \hat{\pi}^{-1} = \hat{\pi} \\ \hat{\pi}^{+} = \hat{\pi} \end{array}\right\}$ $\hat{\pi}$ is unitary

The eigenvectors of $\hat{\pi}$ have even or odd wavefunctions:

$$\psi(-x) = \pm\,\psi(x)$$

$+$ = "even parity"
$-$ = "odd parity"

Since $\psi(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\,\psi(x)$, the p-space wavefunctions satisfy

$$\psi(p) \xrightarrow{\hat{\pi}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}\,\psi(-x) \overset{x'=-x}{=} \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi\hbar}}\, e^{-ipx'}\psi(x')$$

$$= \psi(-p)$$

So $\hat{\pi}|p\rangle = |-p\rangle$ is actually a consequence of $\hat{\pi}|x\rangle = |-x\rangle$, and not an independent property.

In the passive transformation picture

$$\hat{\pi}^{+}\hat{X}\hat{\pi} = -\hat{X}$$

$$\hat{\pi}^{+}\hat{P}\hat{\pi} = -\hat{P}$$

The Hamiltonian is <u>parity invariant</u> if (remember: $\hat{\pi}$ is unitary!)

$$\hat{\pi}^{+}\hat{H}(\hat{X},\hat{P})\hat{\pi} = \hat{H}(\hat{\pi}^{+}\hat{X}\hat{\pi}, \hat{\pi}^{+}\hat{P}\hat{\pi})$$

$$\boxed{= \hat{H}(-\hat{X},-\hat{P}) = \hat{H}(\hat{X},\hat{P})}$$

or $\boxed{[\hat{\pi},\hat{H}] = 0}$

In this case

$$[\hat{\pi}, \hat{U}(t)] = [\hat{\pi}, e^{-i\hat{H}t/\hbar}] = 0$$

for time-independent $\hat{H}$. But this even holds for time-dependent $\hat{H}$ (where energy is not conserved): If $\hat{H}$ is parity invariant at all times, $[\hat{\pi},\hat{H}(t)] = 0 \ \forall t$, then

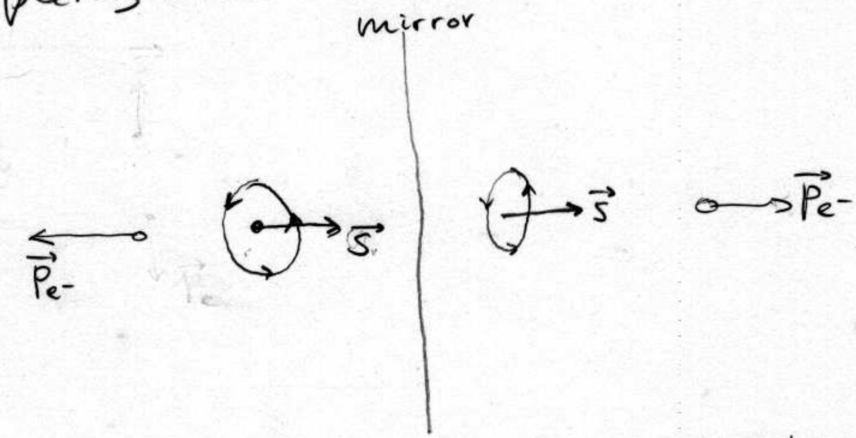
$$[\hat{\pi}, \hat{U}(t)] = [\hat{\pi}, \hat{T} e^{-i/\hbar \int_0^t dt' \hat{H}(t')}] = 0$$

is still zero. So parity is preserved in time if the Hamiltonian is parity invariant.

The weak interaction in the Standard Model of particle physics is not parity invariant. Experimentally parity violation in weak interactions was first observed by Madame Wu in the β-decay of ^{60}Co:

$$^{60}Co \rightarrow {}^{60}Ni + e^- + \bar{\nu}_e \qquad (n \rightarrow p + e^- + \bar{\nu}_e)$$

^{60}Co has a large spin of $5\hbar$. C.S. Wu found in 1957 that the emitted electron preferably came out in a direction opposite to the spin of the Co nuclei. (This had been predicted by T.D. Lee and C.N. Yang in 1956, for which they won the Nobel prize.) This implies parity non-invariance:



In the mirrored experiment the direction $\vec{p}_e$ is flipped, but the ring current representing the orbital motion of the unpaired nucleons generating the spin maintains its orientation; hence now the electron is emitted <u>in</u> the direction of the spin. In Nature only the situation at the left of the mirror, not the one on its right is realized → parity is broken.

11.4 Time-reversal symmetry

If you let time run backwards, $t \to -t$, x is not affected, but $v = \frac{dx}{dt} \to \frac{dx}{d(-t)} = -\frac{dx}{dt}$ flips sign,

and all trajectories $x(t)$ reverse direction:

$$x_{reversed}(t) = x(-t)$$

$$\Rightarrow \quad \dot{x}_{reversed}(t) = \frac{dx(-t)}{dt} = -\frac{dx(-t)}{d(-t)} = -\dot{x}(-t)$$

The reversed trajectory still obeys Newton's law:

$$m \frac{d^2 x_{reversed}(t)}{dt^2} = m \frac{d^2 x(-t)}{dt^2} = m \frac{d^2 x(-t)}{(d(-t))^2} = F(x(-t))$$

$$= F(x_{reversed}(t))$$

Yes, it does!

How does this manifest itself in QM?

S.Eq.: $\quad i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(\hat{X},\hat{P})|\psi(t)\rangle \xrightarrow{x\text{-basis}} i\hbar \frac{\partial \psi(x,t)}{\partial t} = H(x, -i\hbar\frac{\partial}{\partial x})\psi(x,t)$

Take Hermitean adjoint:

$$-i\hbar \frac{d}{dt}\langle\psi(t)| = \langle\psi(t)|\hat{H}^+ = \langle\psi(t)|\hat{H}(\hat{X},\hat{P}) \qquad |\cdot|x\rangle$$

$$\xrightarrow[x\text{-basis}]{} \quad -i\hbar \frac{\partial \psi^*(x,t)}{\partial t} = \langle\psi(t)|\hat{H}(\hat{X},\hat{P})|x\rangle$$

$$= \langle x|\hat{H}(\hat{X},\hat{P})|\psi\rangle^*$$

$$= \left(H(x, -i\hbar\tfrac{\partial}{\partial x})\psi(x,t)\right)^*$$

$$= \left(H(x, -i\hbar\tfrac{\partial}{\partial x})\right)^* \psi^*(x,t)$$

This shows that $\psi^*(x,t)$ satisfies the time-reversed S. Eq.:

$$i\hbar \frac{\partial \psi^*(x,t)}{\partial(-t)} = \underbrace{H(x, +i\hbar\tfrac{\partial}{\partial x})}_{\updownarrow \; \mathcal{H}(x,-p)} \psi^*(x,t)$$

$$\Rightarrow \boxed{\psi_{reversed}(x,t) = \psi^*(x,t)}$$

(this doesn't change the probability $|\psi(x,t)|^2$ at time t; only its time evolution: $|\psi_{rev}(x,t)|^2 = |\psi(x,-t)|^2$)

Time reversal invariance requires

$$\mathcal{H}(x,p) = \mathcal{H}(x,-p) \quad \text{classically}$$

$$\left.\begin{array}{l} \hat{H}(\hat{X},\hat{P}) = \hat{H}(\hat{X},-\hat{P}) \\ H(x,-i\hbar\tfrac{\partial}{\partial x}) = H(x,i\hbar\tfrac{\partial}{\partial x}) = \left(H(x,-i\hbar\tfrac{\partial}{\partial x})\right)^* \end{array}\right\} \text{in QM}$$

x-representation

So $\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}(\hat{X})$ ($V(x)$ real) is time reversal invariant.

However, in the presence of a magnetic field

$$\mathcal{H} = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{r},t)\right)^2}{2m} + q\,\phi(\vec{r},t)$$

$$\xrightarrow{(4.3.7)} \hat{H} = \frac{1}{2m}\left(\hat{\vec{P}}\cdot\hat{\vec{P}} - \frac{q}{c}\left(\hat{\vec{P}}\cdot\hat{\vec{A}} + \hat{\vec{A}}\cdot\hat{\vec{P}}\right) + \frac{q^2}{c^2}\hat{\vec{A}}\cdot\hat{\vec{A}}\right) + q\hat{\phi}$$

$\hat{H}$ has terms linear in $\hat{P}$ that change sign under time reversal, so in x-representation $H(x) \neq H^*(x)$.

To get time-reversal invariance, you must also reverse the direction of $\vec{B} = \vec{\nabla}\times\vec{A}$.