

12. Rolling, Torque, and Angular Momentum

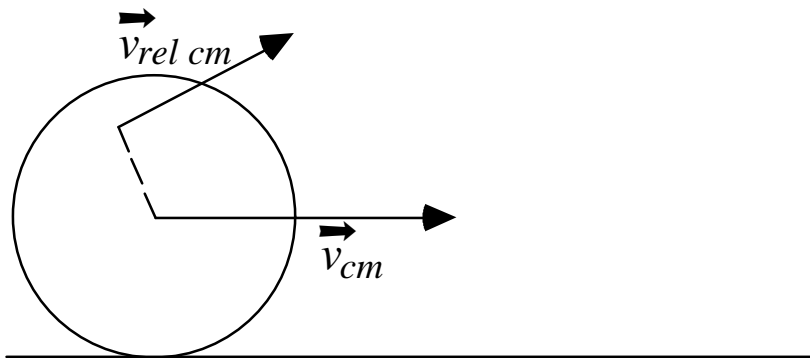
Rolling Motion:

- A motion that is a combination of rotational and translational motion, e.g. a wheel rolling down the road.
- Will only consider rolling with out slipping.

For a disk or sphere rolling along a horizontal surface, the motion can be considered in two ways:

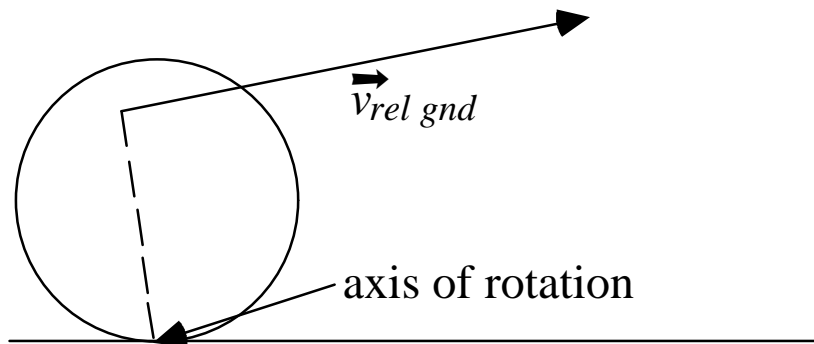
I. Combination of rotational and translational motion:

- Center of mass moves in a translational motion.
- The rest of the body is rotating around the center of mass.

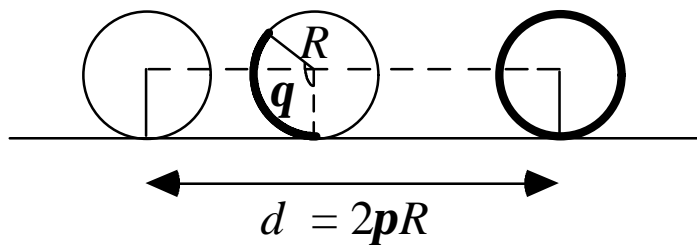


II. Pure Rotational Motion:

- The whole object is revolving around a point on the object in contact with the surface.
- The point of contact changes with time.
- Most people find method I simpler to understand.



Use method I to analyze rolling without slipping:



- When the object makes one complete revolution, the object has moved a distance equal to the circumference, and each point on the exterior has touched the ground once.
- When the object rotates through an angle q , the distance that the center of mass has moved is:

$$s = R\mathbf{q}$$

$$v_{cm} = R \frac{d\mathbf{q}}{dt}$$

$$= R\mathbf{w}$$

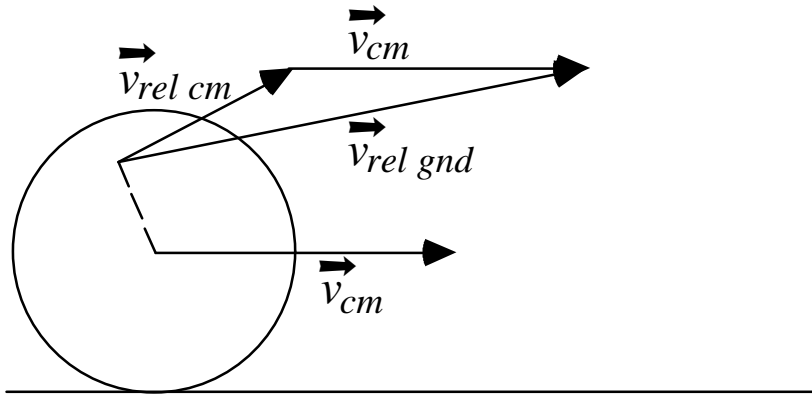
where \mathbf{w} is the angular velocity of one object rotating about its center of mass. This looks very similar to the relationship between angular velocity and the translational velocity of a point on a rotating object:

$$v = R\mathbf{w}$$

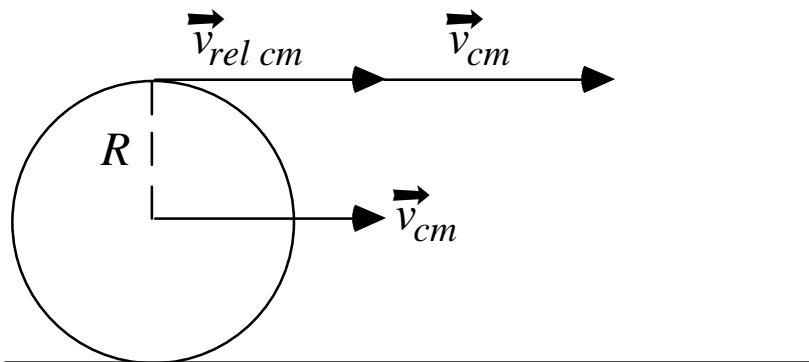
- v_{cm} is the velocity of the center of mass with respect to the ground for the rolling motion.
- v is the velocity of a point on the object with respect to the axis of rotation.

The velocity of any point on the disk as seen by an observer on the ground is the vector sum of the velocity with respect to the center of mass and the velocity of the center of mass with respect to the ground:

$$\dot{\mathbf{v}}_{gnd} = \dot{\mathbf{v}}_{rel\ cm} + \dot{\mathbf{v}}_{cm} \quad (1)$$



Consider the point on the top of the wheel:



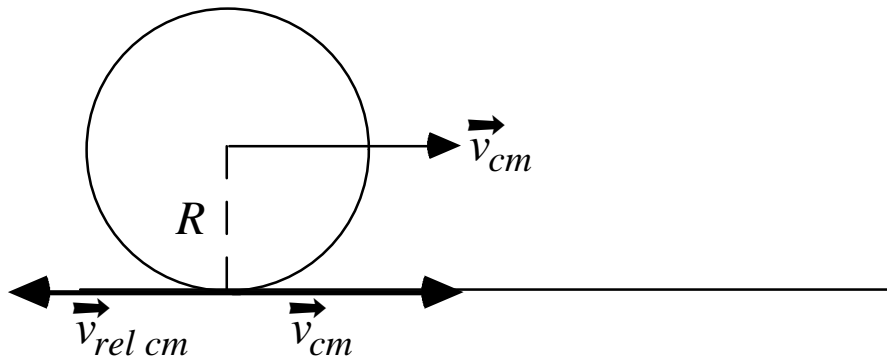
$$v_{rel\ cm} = +R\omega$$

$$v_{cm} = R\omega$$

$$\begin{aligned} (1): \quad v_{gnd} &= R\omega + v_{cm} \\ &= v_{cm} + v_{cm} \\ &= 2v_{cm} \end{aligned}$$

The point on the top of the wheel has a speed (relative to the ground) that is twice the velocity of the center of mass.

Consider the point in contact with the ground:



$$v_{rel\ cm} = -R\omega$$

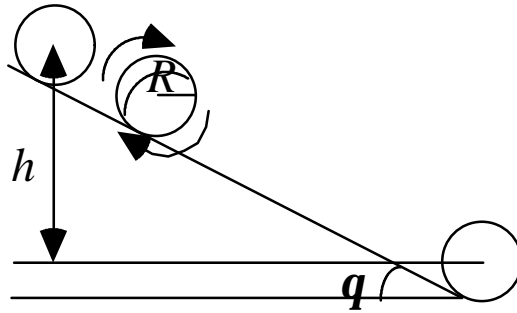
$$v_{cm} = R\omega$$

$$v_{gnd} = -R\omega + R\omega = 0$$

The point in contact with the ground has a speed of zero, i.e. momentarily at rest.

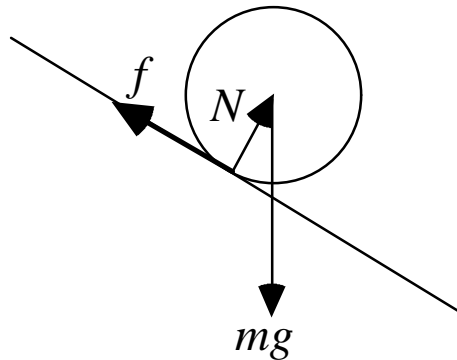
- ★ If your car is traveling down the highway at 70 mph, the tops of your wheels are going 140 mph while the bottoms of the wheels are going 0 mph.

Consider a disk rolling down a ramp without slipping:



Assuming the disk is initially at rest:

- What makes the disk start rolling?
- What is the translational speed of its center of mass when it reaches the bottom of the ramp?
- What is its angular velocity when it reaches the bottom of the ramp?



★ We need to find the torque that makes the disk start rolling. Consider the rotation around the center of mass:

- The force of gravity acts at the center of mass and hence produces no torque.
- The normal force has zero level arm and hence produces no torque.
- The force of friction provides the torque!

- If the object is rolling without slipping, the friction force is static friction.
 - If the ramp is frictionless, the disk will slide down without rotation.
- ★ The velocity and angular velocity at the bottom of the ramp can be calculated using energy conservation. The kinetic energy can be written as a sum of translational and rotational kinetic energy:

$$\begin{aligned}
 K_{tot} &= K_{tran\,cm} + K_{rot\,rel\,to\,cm} \\
 &= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\boldsymbol{w}^2
 \end{aligned}$$

where w is the angular speed of the rotation relative to the center of mass and I_{cm} is the moment of inertia around an axis passing through the center of mass.

Conservation of Energy:

$$E_i = E_f$$

$$mgh = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\boldsymbol{w}^2 \quad (1)$$

Since the disk rolls without slipping:

$$v_{cm} = R\omega$$

$$\omega = \frac{v_{cm}}{R} \quad (2)$$

$$(1): \quad mgh = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2$$

$$= \frac{3}{4}mv_{cm}^2$$

$$v_{cm} = \sqrt{\frac{4gh}{3}}$$

$$(2): \quad \omega = \sqrt{\frac{4gh}{3R^2}}$$

- The speed at the bottom does not depend on the radius or the mass of the disk.
- The speed at the bottom is less than when the disk slides down a frictionless ramp:

$$v = \sqrt{2gh}$$

- The angular speed depends on the radius but not the mass.
- We can still apply conservation of energy even though there is a friction force. The friction force cannot dissipate mechanical energy because it is a static friction. The point

in contact with the surface is at rest while in contact with the surface.

- A sphere or a hollow cylinder will reach the bottom with different speeds due to different centers of mass:

$$E_f^{sphere} = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \left(\frac{2}{5} m R^2 \right) \left(\frac{v_{cm}}{R} \right)^2$$

$$mgh = \frac{7}{10} m v_{cm}^2$$

$$v_{cm} = \sqrt{\frac{10}{7} gh}$$

$$E_f^{hollow cyl} = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \left(m R^2 \right) \left(\frac{v_{cm}}{R} \right)^2$$

$$mgh = m v_{cm}^2$$

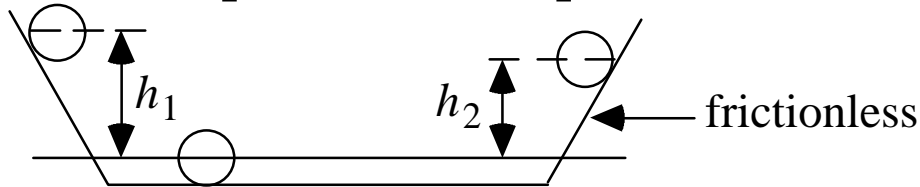
$$v_{cm} = \sqrt{gh}$$

- ★ The sphere will reach the bottom first, followed by the disk and hollow cylinder.
- ★ The result is independent of the mass and radius!

Example:

A sphere rolls down a ramp as shown. The ramp and the floor has friction to keep the ball

rolling without slipping until it reaches the second ramp which is frictionless. How high on the second ramp does the sphere rise?



- The sphere rolls down the 1st ramp and reaches some angular speed at the bottom.
- The sphere rolls across the floor with the same angular speed.
- The sphere slides up the ramp with the same angular speed because there is no torque (friction) acting on the sphere.

$$E_i = mgh_1 \quad (1)$$

$$\begin{aligned} E_{floor} &= \frac{1}{2}mv^2 + I\omega^2 \\ &= \frac{1}{2}m\omega^2 R^2 + \frac{1}{2} \times \frac{2}{5}mR^2 \times \omega^2 \\ &= \frac{7}{10}m\omega^2 R^2 \end{aligned} \quad (2)$$

$$\begin{aligned} E_f &= mgh_2 + \frac{1}{2}I\omega^2 \\ &= mgh_2 + \frac{1}{2} \times \frac{2}{5}mR^2 \times \omega^2 \\ &= mgh_2 + \frac{1}{5}m\omega^2 R^2 \end{aligned} \quad (3)$$

Apply conservation of energy to (1) and (2):

$$mgh_1 = \frac{7}{10} m\omega^2 R^2$$

$$\omega^2 R^2 = \frac{10}{7} gh_1 \quad (4)$$

Apply conservation of energy to (1) and (3):

$$mgh_1 = mgh_2 + \frac{1}{5} m\omega^2 R^2$$

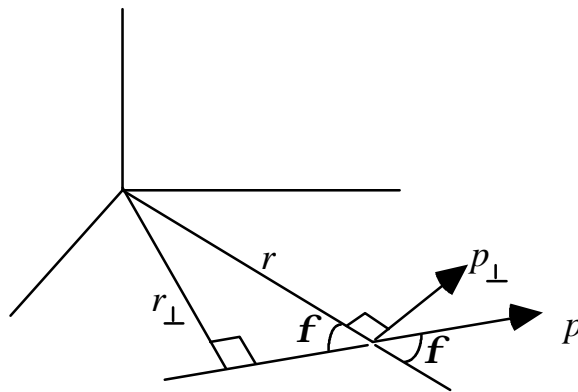
$$gh_1 = gh_2 + \frac{1}{5} \omega^2 R^2$$

$$(4): \quad gh_1 = gh_2 + \frac{2}{7} gh_1$$

$$h_2 = \frac{5}{7} h_1$$

Angular Momentum:

- rotational analogue of linear momentum
- must be defined with respect to some point



- similar to torque, two calculation methods:

$$L = rp_{\perp} = rp \sin f = rmv \sin f$$

$$L = r_{\perp}p = (r \sin f)p = rmv \sin f$$

- A vector quantity:

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \wedge \dot{\mathbf{p}}$$

★ $\dot{\mathbf{L}}$ is perpendicular to $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$

★ direction is defined by the right hand rule

- unit: $\text{kg} \cdot \text{m}^2 / \text{s}$
- A particle does not have to travel in a circle to have angular momentum. A particle traveling in a straight line has angular momentum relative to a particular point.
- If a point that the angular momentum is defined relative to is along the momentum vector, then the angular momentum is zero ($r_{\perp} = 0$).
- For a system of particles:

$$\dot{\mathbf{L}} = \sum \dot{\mathbf{L}}_i = \dot{\mathbf{L}}_1 + \dot{\mathbf{L}}_2 + \dots + \dot{\mathbf{L}}_n$$

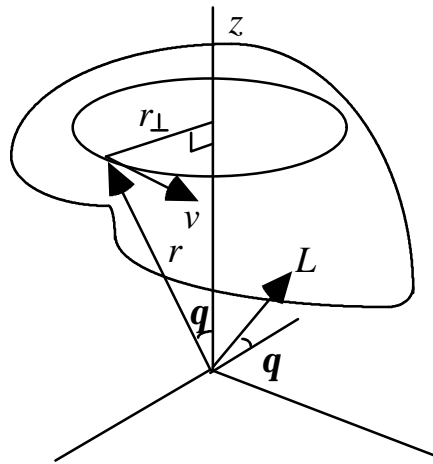
Newton's Second Law:

Translational: $\sum \mathbf{F}_{ext} = \frac{d\mathbf{p}}{dt}$

Rotational: $\sum \mathbf{\tau}_{ext} = \frac{d\mathbf{L}}{dt}$

- The vector sum of all the torques acting on a particle is equal to the rate of change of the angular momentum.
- Both t and L must be defined relative to the same origin.

Angular Momentum around a Fixed Axis:



Angular momentum of a small element of mass dm on the object:

$$dL_z = dL \sin \mathbf{q}$$

$$= r(dm)v \sin \mathbf{q}$$

$$= r_{\perp}(dm)v$$

$$= \mathbf{w}r_{\perp}^2 dm$$

$$L_z = \mathbf{w} \int r_{\perp}^2 dm$$

$$L_z = I\mathbf{w}$$

The component of the angular momentum along a fixed axis of rotation is just the moment of inertia times the angular velocity.

We often just write:

$$L = I\omega$$

- Other components often cancel by symmetry

Conservation of Angular Momentum:

- If any component of the net external torque on a system is zero, then the component of the angular momentum of the system along that axis is conserved.
- If a rotating object can somehow change its moment of inertia by internal forces, then the object will spin faster or slower depending on whether the moment of inertia decreases or increases:

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

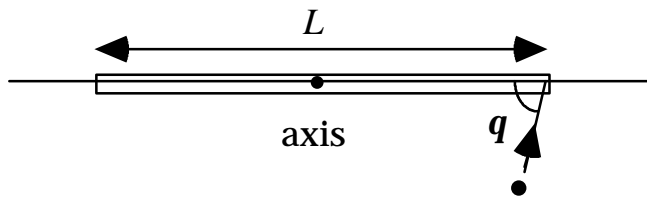
$$\omega_f = \frac{I_i}{I_f} \omega_i$$

Example:

An ice skater would start spinning with their arms extended away from the center of her body (the axis of rotation). As the ice skater pulls her arms tight to her body, the mass is now closer to the axis of rotation, therefore the moment of inertia has been reduced and the skater spins faster in order to conserve angular momentum.

Example:

A uniform thin rod of length 0.5 m and mass 4.0 kg can rotate in a horizontal plane about a vertical axis through its center. The rod is at rest when a 3.0 g bullet traveling the horizontal plane is fired into one end of the rod. As viewed from above, the direction of the bullet's velocity makes an angle of 60° with the rod. If the bullet lodges in the rod and the angular velocity of the rod is 10 rad/s immediately after the collision, what is the magnitude of the bullet's velocity just before the impact?



Conservation of angular momentum:

$$L_i = L_f$$

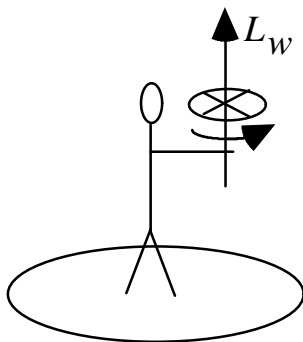
$$m_b v_b \frac{L}{2} \sin \mathbf{q} = (I_r + I_b) \mathbf{w}_f$$

$$= \left[\frac{1}{12} m_r L^2 + m_b \left(\frac{L}{2} \right)^2 \right] \mathbf{w}_f$$

$$\mathbf{w}_f = \frac{m_b v_b \frac{L}{2} \sin \mathbf{q}}{\frac{1}{12} m_r L^2 + m_b \left(\frac{L}{2} \right)^2} = 1290 \text{ m / s}$$

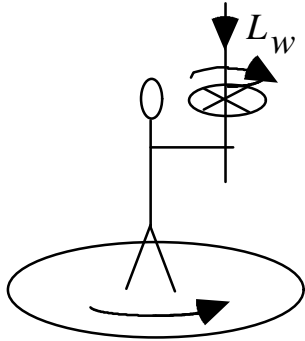
Example:

Consider a person standing on a platform that can rotate. The person is holding a wheel that is spinning such that its angular momentum is pointing upwards:



$$L_i = L_w$$

If the person turns the wheel over so that the angular momentum of the wheel is pointing down, what happens to the motion of the man?



The person must start rotating because the angular momentum of the person-wheel system must be conserved:

$$L_f = -L_w + L_p$$

$$L_w = -L_w + L_p$$

$$L_p = 2L_w$$