Introduction:

Understanding of many physical phenomena depend on statistical and probabilistic concepts:

- Statistical Mechanics (physics of systems composed of many parts: gases, liquids, solids.)
  - 1 mole of anything contains $6 \times 10^{23}$ particles (Avogadro's number)
  - impossible to keep track of all $6 \times 10^{23}$ particles even with the fastest computer imaginable
  - resort to learning about the group properties of all the particles
  - partition function: calculate energy, entropy, pressure... of a system

- Quantum Mechanics (physics at the atomic or smaller scale)
  - wavefunction = probability amplitude
  - probability of an electron being located at $(x,y,z)$ at a certain time.

Understanding/interpretation of experimental data depend on statistical and probabilistic concepts:

- how do we extract the best value of a quantity from a set of measurements?
- how do we decide if our experiment is consistent/inconsistent with a given theory?
- how do we decide if our experiment is internally consistent?
- how do we decide if our experiment is consistent with other experiments?
  - In this course we will concentrate on the above experimental issues!
Definition of probability:

- Suppose we have $N$ trials and a specified event occurs $r$ times.
  - example: rolling a dice and the event could be rolling a 6.
- define probability ($P$) of an event ($E$) occurring as:
  \[ P(E) = \frac{r}{N} \text{ when } N \rightarrow \infty \]
- examples:
  - six sided dice: $P(6) = \frac{1}{6}$
  - coin toss: $P($heads$) = 0.5$
    - $P($heads$)$ should approach 0.5 the more times you toss the coin.
    - for a single coin toss we can never get $P($heads$) = 0.5$!
- by definition probability is a non-negative real number bounded by $0 \leq P \leq 1$
  - if $P = 0$ then the event never occurs
  - if $P = 1$ then the event always occurs
- sum (or integral) of all probabilities if they are mutually exclusive must = 1.
  - events are independent if: $P(A \cap B) = P(A)P(B)$
  - events are mutually exclusive (disjoint) if: $P(A \cap B) = 0$ or $P(A \cup B) = P(A) + P(B)$
Probability can be a discrete or a continuous variable.

Discrete probability: $P$ can have certain values only.

- examples:
  - tossing a six-sided dice: $P(x_i) = P_i$ here $x_i = 1, 2, 3, 4, 5, 6$ and $P_i = 1/6$ for all $x_i$.
  - tossing a coin: only 2 choices, heads or tails.

for both of the above discrete examples (and in general) when we sum over all mutually exclusive possibilities:
\[ \sum_i P(x_i) = 1 \]

Continuous probability: $P$ can be any number between 0 and 1.

- define a “probability density function”, pdf, $f(x)$
\[ f(x)dx = dP(x \leq x + dx) \quad \text{with } x \text{ a continuous variable} \]
- probability for $x$ to be in the range $a \leq x \leq b$ is:
\[ P(a \leq x \leq b) = \int_a^b f(x)dx \]

just like the discrete case the sum of all probabilities must equal 1.
\[ \int \int f(x)dx = 1 \]

$f(x)$ is normalized to one.

probability for $x$ to be exactly some number is zero since:
\[ \int_a^a f(x)dx = 0 \]
Examples of some common $P(x)$’s and $f(x)$’s:

**Discrete = $P(x)$**  **Continuous = $f(x)$**
- binomial
- Poisson
- uniform, i.e. constant
- Gaussian
- exponential
- chi square

How do we describe a probability distribution?
- mean, mode, median, and variance
- for a continuous distribution, these quantities are defined by:

<table>
<thead>
<tr>
<th>Mean</th>
<th>Mode</th>
<th>Median</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>average</td>
<td>most probable</td>
<td>50% point</td>
<td>width of distribution</td>
</tr>
</tbody>
</table>

$$\bar{\mu} = \int f(x) \, dx$$

$$\frac{\partial f(x)}{\partial x} \bigg|_{x=a} = 0$$

$$0.5 = \int f(x) \, dx$$

$$\sigma^2 = \int f(x)(x - \mu)^2 \, dx$$

for a discrete distribution, the mean and variance are defined by:

$$\bar{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$
Some continuous pdf:

For a Gaussian pdf, the mean, mode, and median are all at the same $x$.

For most pdfs, the mean, mode, and median are at different locations.
Calculation of mean and variance:

- example: a discrete data set consisting of three numbers: \( \{1, 2, 3\} \)
  - average \(( \bar{x} )\) is just:
    \[
    \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1 + 2 + 3}{3} = 2
    \]
  - complication: suppose some measurement are more precise than others.
    - if each measurement \( x_i \) have a weight \( w_i \) associated with it:
      \[
      \bar{x} = \frac{1}{\sum_{i=1}^{n} w_i} \sum_{i=1}^{n} x_i w_i
      \]
      “weighted average”
  - variance \(( \sigma^2 )\) or average squared deviation from the mean is just:
    \[
    \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2
    \]
    variance describes the width of the pdf!
  - \( \sigma \) is called the standard deviation
  - rewrite the above expression by expanding the summations:
    \[
    \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 + \frac{1}{n} \left( \sum_{i=1}^{n} x_i \right)^2 - 2 \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i
    \]
    \[
    = \frac{1}{n} \sum_{i=1}^{n} x_i^2 + \sigma^2 - 2 \sigma^2
    \]
    \[
    = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \sigma^2
    \]
    \[
    = \langle x^2 \rangle - \langle x \rangle^2
    \]
    \[< > \equiv \text{average} \]
  - \( n \) in the denominator would be \( n - 1 \) if we determined the average \(( \bar{x} )\) from the data itself.
using the definition of $\mu$ from above we have for our example of \{1,2,3\}:

$$\mu^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \mu^2 = 4.67 \mu^2 = 0.67$$

the case where the measurements have different weights is more complicated:

$$\mu^2 = \sum_{i=1}^{n} w_i (x_i - \mu)^2 / \sum_{i=1}^{n} w_i^2 \mu^2 = \sum_{i=1}^{n} w_i x_i^2 / \sum_{i=1}^{n} w_i$$

- $\mu$ is the weighted mean
- if we calculated $\mu$ from the data, $\mu^2$ gets multiplied by a factor $n/(n-1)$.

example: a continuous probability distribution, $f(x) = \sin^2 x$ for $0 \leq x \leq 2\pi$

- has two modes!
- has same mean and median, but differ from the mode(s).

$f(x)$ is not properly normalized: $\int_0^{2\pi} \sin^2 x \, dx = \mu \neq 1$

normalized pdf: $f(x) = \sin^2 x / \int_0^{2\pi} \sin^2 x \, dx = \frac{1}{\mu} \sin^2 x$
for continuous probability distributions, the mean, mode, and median are calculated using either integrals or derivatives:

\[
\mu = \frac{1}{2} \int_0^\infty x \sin^2 x \, dx = \mu
\]

mode: \[ \frac{\partial}{\partial x} \sin^2 x = 0 \quad \mu = \frac{3}{2}, \frac{3}{2} \]

median: \[ \frac{1}{2} \int_0^a \sin^2 x \, dx = \frac{1}{2} \quad a = \mu \]

example: Gaussian distribution function, a continuous probability distribution

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \text{gaussian} \]

\[ \sigma = \text{standard deviation} \]

68% of area within \( \pm 1\sigma \)
Accuracy and Precision:

- Accuracy: The accuracy of an experiment refers to how close the experimental measurement is to the true value of the quantity being measured.
- Precision: This refers to how well the experimental result has been determined, without regard to the true value of the quantity being measured.

Just because an experiment is precise it does not mean it is accurate!!

Measurements of the neutron lifetime over the years:

![Diagram showing steady increase in precision but any of these measurements accurate?](image)

The size of bar reflects the precision of the experiment.
Measurement Errors (Uncertainties)
- Use results from probability and statistics as a way of indicating how “good” a measurement is.
  - most common quality indicator:
    relative precision = [uncertainty of measurement]/measurement
  - example: we measure a table to be 10 inches with uncertainty of 1 inch.
    relative precision = 1/10 = 0.1 or 10% (% relative precision)
- uncertainty in measurement is usually square root of variance:
  \( \sigma = \text{standard deviation} \)
  - usually calculated using the technique of “propagation of errors”.

Statistics and Systematic Errors
- Results from experiments are often presented as:
  \( N \pm XX \pm YY \)
  - \( N \): value of quantity measured (or determined) by experiment.
  - \( XX \): statistical error, usually assumed to be from a Gaussian distribution.
    - With the assumption of Gaussian statistics we can say (calculate) something about how well our experiment agrees with other experiments and/or theories.
    - Expect an 68% chance that the true value is between \( N - XX \) and \( N + XX \).
  - \( YY \): systematic error. Hard to estimate, distribution of errors usually not known.
- examples: mass of proton = 0.9382769 ± 0.0000027 GeV
  - mass of W boson = 80.8 ± 1.5 ± 2.4 GeV
What’s the difference between statistical and systematic errors?
- statistical errors are “random” in the sense that if we repeat the measurement enough times: $XX \to 0$
- systematic errors do not $\to 0$ with repetition.
  - examples of sources of systematic errors:
    - voltmeter not calibrated properly
    - a ruler not the length we think is (meter stick might really be < meter!)
- because of systematic errors, an experimental result can be precise, but not accurate!

How do we combine systematic and statistical errors to get one estimate of precision?
- big problem!
- two choices:
  - $\sigma_{tot} = XX + YY$ add them linearly
  - $\sigma_{tot} = (XX^2 + YY^2)^{1/2}$ add them in quadrature

Some other ways of quoting experimental results
- lower limit: “the mass of particle $X$ is > 100 GeV”
- upper limit: “the mass of particle $X$ is < 100 GeV”
- asymmetric errors: mass of particle $X = 100^{+4}_{-3}$ GeV