RLC series circuit:

- What's $V_R$?
  - Simplest way to solve for $V$ is to use voltage divider equation in complex notation:

$$V_R = \frac{V_{in}R}{R + X_C + X_L} = \frac{V_{in}R}{R + \frac{1}{j\omega C} + j\omega L}$$

- Using complex notation for the apply voltage $V_{in} = V_0\cos\omega t = \text{Real}(V_0e^{j\omega t})$:

$$V_R = \frac{V_0e^{j\omega t}R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

  - We are interested in the both the magnitude of $V_R$ and its phase with respect to $V_{in}$.
  - First the magnitude:

$$|V_R| = \frac{|V_0e^{j\omega t}|R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{V_0R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$
The phase of $V_R$ with respect to $V_{in}$ can be found by writing $V_R$ in purely polar notation. For the denominator we have:

$$R + j \left( \omega L - \frac{1}{\omega C} \right) = \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2} \exp \left\{ j \tan^{-1} \left[ \frac{\omega L - \frac{1}{\omega C}}{R} \right] \right\}$$

Define the phase angle $\phi$:

$$\tan \phi = \frac{\text{Imaginary } X}{\text{Real } X} = \frac{\omega L - \frac{1}{\omega C}}{R} = \frac{R}{\omega C}$$

We can now write for $V_R$ in complex form:

$$V_R = \frac{V_0 R e^{j\omega t}}{e^{j\phi} \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}$$

Depending on $L$, $C$, and $\omega$, the phase angle can be positive or negative! In this example, if $\omega L > 1/\omega C$, then $V_R(t)$ lags $V_{in}(t)$.

Finally, we can write down the solution for $V$ by taking the real part of the above equation:

$$V_R = \text{Real} \left\{ \frac{V_0 R e^{j(\omega t - \phi)}}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}} \right\} = \frac{V_0 R \cos(\omega t - \phi)}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}}$$
$R = 100 \ \Omega, \ L = 0.1 \ \text{H}, \ C = 0.1 \ \mu\text{F}$

- $V_R \ll V_{in}$ at 100 Hz.
- $V_R$ and $V_{in}$ are not in phase at this frequency.
- The little wiggles on $V_R$ are real!
  - Transient solution (homogeneous solution) to the differential eq. describing the circuit.
  - After a few cycles this contribution to $V_R$ die out.
Bode plot of magnitude of $V_R/V_{in}$ vs. frequency
In general $V_C(t)$, $V_R(t)$, and $V_L(t)$ are all out of phase with the applied voltage.

$I(t)$ and $V_R(t)$ are in phase in a series RLC circuit.

The amplitude of $V_C$, $V_R$, and $V_L$ depend on $\omega$.

The table below summarizes the 3 cases with the following definitions:

\[ Z = \left[ R^2 + (\omega L - 1/\omega C)^2 \right]^{1/2} \]

\[ \tan \phi = (\omega L - 1/\omega C)/R \]

<table>
<thead>
<tr>
<th>Gain</th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_R/V_{in}$</td>
<td>$R/Z$</td>
<td>$-\phi$</td>
</tr>
<tr>
<td>$V_L/V_{in}$</td>
<td>$\omega L/Z$</td>
<td>$\pi/2 - \phi$</td>
</tr>
<tr>
<td>$V_C/V_{in}$</td>
<td>$1/\omega CZ$</td>
<td>$-\pi/2 - \phi$</td>
</tr>
</tbody>
</table>

RLC circuits are resonant circuits

- energy in the system “resonates” between the inductor and capacitor
- “ideal” capacitors and inductors do not dissipate energy
- resistors dissipate energy i.e. resistors do not store energy
• Resonant Frequency:
  ◆ At the resonant frequency the imaginary part of the impedance vanishes.
  ◆ For the series RLC circuit the impedance \( Z \) is:
    \[
    Z = R + X_L + X_C = R + j(\omega L - 1/\omega C)
    \]
    \[
    |Z| = \left[ R^2 + (\omega L - 1/\omega C)^2 \right]^{1/2}
    \]
  ◆ At resonance (series, parallel etc):
    \[
    \omega L = 1/\omega C
    \]
    \[
    \omega_R = \frac{1}{\sqrt{LC}}
    \]
  ◆ At the resonant frequency the following are true for a series RLC circuit:
    ■ \( |V_R| \) is maximum (ideally = \( V_{in} \))
    ■ \( \phi = 0 \)
    ■ \[
    \frac{|V_C|}{|V_{in}|} = \frac{|V_L|}{|V_{in}|} = \frac{\sqrt{L}}{R\sqrt{C}} \quad (V_C \text{ or } V_L \text{ can be } > V_{in}!)
    \]
    ☞ The circuit acts like a narrow band pass filter.

• There is an exact analogy between an RLC circuit and a harmonic oscillator (mass attached to spring):
  ◆ \[
  m \frac{d^2x}{dt^2} + B \frac{dx}{dt} + kx = 0 \quad \text{damped harmonic oscillator}
  \]
  ◆ \[
  L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{undriven RLC circuit}
  \]
  ◆ \( x \leftrightarrow q \) (electric charge), \( L \leftrightarrow m, \quad k \leftrightarrow 1/C \)
  ◆ \( B \) (coefficient of damping) \( \leftrightarrow R \)
- Q (quality factor) of a circuit: determines how well the RLC circuit stores energy
  - $Q = 2\pi \frac{\text{max energy stored}}{\text{energy lost}}$ per cycle
  - Q is related to sharpness of the resonance peak:
The maximum energy stored in the inductor is \( LI^2/2 \) with \( I = I_{\text{MAX}} \).

- no energy is stored in the capacitor at this instant because \( I \) and \( V_C \) are 90\(^0\) out of phase.
- The energy lost in one cycle:

\[
\text{power} \times \text{(time for cycle)} = I_{\text{RMS}}^2 R \times \frac{2\pi}{\omega_R} = \frac{1}{2} I_{\text{MAX}}^2 R \times \frac{2\pi}{\omega_R}
\]

\[
Q = \frac{2\pi \left( \frac{LI_{\text{MAX}}^2}{2} \right)}{2\pi \left( \frac{RI_{\text{MAX}}^2}{2} \right)} = \frac{\omega_R L}{R}
\]

- There is another popular, equivalent expression for \( Q \)

\[
Q = \frac{\omega_R}{\omega_U - \omega_L}
\]

- \( \omega_U (\omega_L) \) is the upper (lower) 3 dB frequency of the resonance curve.
  - \( Q \) is related to sharpness of the resonance peak.
- Will skip the derivation here as it involves a bit of algebra.
  - two crucial points of the derivation:

\[
\frac{V_R}{V_{\text{in}}} = \frac{1}{\sqrt{1 + Q^2 \left( \frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right)^2}}
\]

\[
Q \left( \frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right) = \pm 1 \quad \text{at the upper and lower 3 dB points}
\]
● $Q$ can be measured from the shape of the resonance curve
  ■ one does not need to know $R$, $L$, or $C$ to find $Q$!

\[
Q = \frac{\omega_R}{\omega_U - \omega_L}
\]

Example: Audio filter (band pass filter)
  ■ Audio filter is matched to the frequency range of the ear (20-20,000 Hz).
Let's design an audio filter using low and high pass RC circuits.

\[
\begin{align*}
\text{Vin} & \quad R_1 \quad C_1 \quad Z_1 \\
\text{stage 1} & \\
\text{R}_2 & \quad C_2 \quad \text{Z}_2 \\
\quad & \quad \text{Vout} \\
\text{stage 2} & \\
\end{align*}
\]

Ideally, the frequency response is flat over 20-20,000 Hz, and rolls off sharply at frequencies below 20 Hz and above 20,000 Hz.

- Set 3 dB points as follows:
  - lower 3 dB point : 20 Hz = \(1/2\pi R_1 C_1\)
  - upper 3 dB point: 2x10^4 Hz = \(1/2\pi R_2 C_2\)

- If we put these two filters together we don't want the 2\(^{nd}\) stage to affect the 1\(^{st}\) stage.
  - can accomplish this by making the impedance of the 2\(^{nd}\) \(Z_2\) stage much larger than \(R_1\).
  - Remember \(R_1\) is in parallel with \(Z_2\).

\[
\begin{align*}
Z_1 &= R_1 + 1/j\omega C_1 \\
Z_2 &= R_2 + 1/j\omega C_2
\end{align*}
\]

- In order to insure that the second stage does not “load” down the first stage we need: \(R_2 \gg R_1\) since at high frequencies \(Z_2 \Rightarrow R_2\)

- We can now pick and calculate values for the \(R\)'s and \(C\)'s in the problem.
  - Let \(C_1 = 1 \mu F \Rightarrow R_1 = 1/(20\text{Hz} \cdot 2\pi C_1) = 8 \text{ k}\Omega\)
  - Let \(R_2 > 100R_1 \Rightarrow R_2 = 1 \text{ M}\Omega\), and \(C_2 = 1/(2x10^4 \text{ Hz} \cdot 2\pi R_2) = 8 \text{ pf}\)
  - \(R_1 = 8 \text{ k}\Omega, C_1 = 1 \mu F\)
  - \(R_2 = 1 \text{ M}\Omega, C_2 = 8 \text{ pf}\)
Exact derivation for above filter:

- In the above circuit we treated the two RC filters as independent.
- Why did this work?
- We want to calculate the gain ($|V_{\text{out}}/V_{\text{in}}|$) of the following circuit:

![Diagram of the circuit]

Working from right to left, we have:

- $V_{\text{out}} = V_a X_2 / (X_2 + R_2)$
- $V_a = V_{\text{in}} Z_1 / Z_T$
- $Z_T$ is the total impedance of the circuit as seen from the input.
- $Z_1$ is the parallel impedance of $R_1$ and $R_2$, in series with $C_2$.

$$Z_1 = \frac{R_1 (R_2 + X_2)}{R_1 + R_2 + X_2}$$

$$Z_T = X_1 + Z_1$$

$$V_a = \frac{V_{\text{in}} R_1 (R_2 + X_2)}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)}$$

Finally we can solve for the gain $G = |V_{\text{out}}/V_{\text{in}}|$:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_1 X_2}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)}$$
- We can relate this to our previous result by rewriting the above as:

\[
\frac{V_{out}}{V_{in}} = \frac{R_1 \frac{X_2}{R_2 + X_2}}{X_1 \left( \frac{R_1}{R_2 + X_2} + 1 \right) + R_1}
\]

- If we now remember the approximation \((R_1 << R_2 + X_2)\) made on the previous page to insure that the second stage did not load down the first then we get the following:

\[
\frac{V_{out}}{V_{in}} = \frac{R_1 \frac{X_2}{R_2 + X_2}}{R_1 + X_1 \frac{R_1}{R_2 + X_2}}
\]

- The gain of the circuit looks like the product of two filters, one high pass and one low pass!

- If we calculate the gain of this circuit in dB, the total gain is the sum of the gain of each piece:

\[
\text{Gain in dB} = 20 \log \left( \frac{V_{out}}{V_{in}} \right)
\]

\[
= 20 \log \left( \frac{R_1}{R_1 + X_1} \right) + 20 \log \left( \frac{X_2}{R_2 + X_2} \right)
\]

- The gain of successive filters measured in dB's add!
Another Example: Calculate \( |I| \) and the phase angle between \( V_{in} \) and \( I \) for the following circuit:

\[ V = 230 \text{V RMS} \]

- First calculate \( |I| \).
  - The total current out of the input source \( (I) \) is related to \( V_{in} \) and the total impedance \( (Z_T) \) of the circuit by Ohm’s law:
    \[
    I = \frac{V_{in}}{Z_T}
    \]
  - The total impedance of the circuit is given by the parallel impedance of the two branches:
    \[
    \frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2}
    \]
    \[
    Z_1 = R_1 + X_1
    \]
    \[
    Z_2 = R_2 + X_2
    \]
  - Putting in numerical values for the \( R \)'s and \( X \)'s we have:
    \[
    Z_1 = 20 + j37.7 \ \Omega
    \]
    \[
    Z_2 = 10 - j53.1 \ \Omega
    \]
    \[
    Z_T = 67.4 + j11.8 \ \Omega
    \]
  - We can now find the magnitude of the current:
    \[
    |I| = \frac{|V_{in}|}{|Z_T|}
    \]
    \[
    = \frac{230 \text{ V}}{68.4 \ \Omega}
    \]
    \[
    = 3.36 \text{ A}
    \]
    This is RMS value since \( |V_{in}| \) is given as RMS

K.K. Gan
L4: RLC and Resonance Circuits
Calculate the phase angle between $V_{in}$ and $I$:

- It’s easiest to solve this by writing $V$ and $Z$ in polar form:
  
  
  $V_{in} = (230) e^{j \omega t}$

  $Z_T = (68.4) e^{j \phi}$

  $\tan \phi = \text{Im} Z_T / \text{Re} Z_T$

  \[ = \frac{11.8}{67.4} \]

  $\phi = 9.9^0$

- Finally we can write for the current:

  $I = 3.36 e^{j (\omega t - \phi)}$

- Taking the real part of $I$:

  $I = 3.36 \cos(\omega t - 9.9^0) \ A$

  The current lags the voltage by $9.9^0$. 