Lecture 3: R-L-C AC Circuits

AC (Alternative Current):

- Most of the time, we are interested in the voltage at a point in the circuit.
- We will concentrate on voltages here rather than currents.
- We encounter AC circuits whenever a periodic voltage is applied to a circuit.
- The most common periodic voltage is in the form of a sine (or cosine) wave:
  \[ V(t) = V_0 \cos \omega t \quad \text{or} \quad V(t) = V_0 \sin \omega t \]

- \( V_0 \) is the **amplitude**:
  - \( V_0 = \text{Peak Voltage (} V_p \text{)} \)
  - \( V_0 = 1/2 \text{ Peak-to-Peak Voltage (} V_{PP} \text{)} \)
    - \( V_{PP} \): easiest to read off scope
  - \( V_0 = \sqrt{2} V_{RMS} = 1.41 \ V_{RMS} \)
    - \( V_{RMS} \): what multimeters usually read
    - Multimeters also usually measure the RMS current
- \( \omega \) is the **angular frequency**:
  - \( \omega = 2\pi f \), with \( f \) = frequency of the waveform.
  - frequency (\( f \)) and period (\( T \)) are related by:
    \[ T \text{ (sec)} = \frac{1}{f \text{ (sec}^{-1})} \]
- **Household line voltage** is usually 110-120 \( V_{RMS} \) (156-170 \( V_p \)), \( f = 60 \text{ Hz} \).
  
- It is extremely important to be able to analyze circuits (systems) with sine or cosine inputs
  - Almost any waveform can be constructed from a sum of sines and cosines.
  - This is the “heart” of *Fourier analysis* (Simpson, Chapter 3).
  - The response of a circuit to a complicated waveform (e.g. a square wave) can be understood by analyzing individual sine or cosine components that make up the complicated waveform.
  - Usually only the first few components are important in determining the circuit’s response to the input waveform.

**R-C Circuits and AC waveforms**
- There are many different techniques for solving AC circuits
  - All are based on Kirchhoff’s laws.
  - When solving for voltage and/or current in an AC circuit we are really solving a differential eq.
  - Different circuit techniques are really just different ways of solving the same differential eq:
    - brute force solution to differential equation
    - complex numbers (algebra)
    - Laplace transforms (integrals)
We will solve the following RC circuit using the brute force method and complex numbers method.

Let the input (driving) voltage be \( V(t) = V_0 \cos \omega t \) and we want to find \( V_R(t) \) and \( V_C(t) \).

**Brute Force Method:** Start with Kirchhoff's loop law:

\[
V(t) = V_R(t) + V_C(t)
\]

\[
V_0 \cos \omega t = IR + Q/C
\]

\[
= RdQ(t)/dt + Q(t)/C
\]

We have to solve an inhomogeneous D.E.

The usual way to solve such a D.E. is to assume the solution has the same form as the input:

\[
Q(t) = \alpha \sin \omega t + \beta \cos \omega t
\]

Plug our trial solution \( Q(t) \) back into the D.E.:

\[
V_0 \cos \omega t = aR\omega \cos \omega t - bR\omega \sin \omega t + (\alpha/C) \sin \omega t + (\beta/C) \cos \omega t
\]

\[
= (aR\omega + \beta/C) \cos \omega t + (\alpha/C - bR\omega) \sin \omega t
\]

\[
V_0 = aR\omega + \beta/C
\]

\[
\alpha/C = bR\omega
\]

\[
\alpha = \frac{RC^2 \omega V_0}{1+(RC\omega)^2}
\]

\[
\beta = \frac{CV_0}{1+(RC\omega)^2}
\]
We can now write the solution for $V_C(t)$:

$$V_C(t) = \frac{Q}{C} = (\alpha \sin \omega t + \beta \cos \omega t) / C$$

$$= \frac{RC\omega V_0}{1 + (RC\omega)^2} \sin \omega t + \frac{V_0}{1 + (RC\omega)^2} \cos \omega t$$

We would like to rewrite the above solution in such a way that only a cosine term appears.

- In this form we can compare it to the input voltage.

$$V_C(t) = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \left[ \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}} \sin \omega t + \frac{1}{\sqrt{1 + (RC\omega)^2}} \cos \omega t \right]$$

- We get the above equation in terms of cosine only using the following basic trig:

$$\cos(\theta_1 - \theta_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2$$

- We can now define an angle such that:

$$\cos \phi = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

$$\sin \phi = \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}}$$

$$\tan \phi = RC\omega$$

$$V_C(t) = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \cos(\omega t - \phi)$$

- $V_C(t)$ and $V_0(t)$ are out of phase.
Using the above expression for $V_C(t)$, we obtain:

$$V_R(t) = IR$$

$$= R \frac{dQ}{dt}$$

$$= RC \frac{dV_C}{dt}$$

$$= \frac{-RC \omega V_o}{\sqrt{1 + (RC \omega)^2}} \sin(\omega t - \phi)$$

We would like to have cosines instead of sines by using:

$$-\sin \theta = \cos(\theta + \frac{\pi}{2})$$

$$V_R(t) = \frac{RC \omega V_o}{\sqrt{1 + (RC \omega)^2}} \cos(\omega t - \phi + \frac{\pi}{2})$$

- $V_C(t)$, $V_R(t)$, and $I(t)$ are all out of phase with the applied voltage.
- $I(t)$ and $V_R(t)$ are in phase with each other.
- $V_C(t)$ and $V_R(t)$ are out of phase by $90^\circ$.
- The amplitude of $V_C(t)$ and $V_R(t)$ depend on $\omega$. 
Example: RC Circuit

![Diagram of RC Circuit]

- **R1**: 1E3Ω
- **C2**: 1E-5F
- **Vs**: 60 Hz
- **Vp**: 1 V
- **Vout**

![Waveform Graphs]

- **V_in**
- **V_out**

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L3: R-L-C AC Circuits
Solving circuits with complex numbers:

**PROS:**
- don't explicitly solve differential equations (lots of algebra).
- can find magnitude and phase of voltage separately.

**CONS:**
- have to use complex numbers!
- No “physics” in complex numbers.

What's a complex number? (see Simpson, Appendix E, P835)

- Start with \( j = \sqrt{-1} \) (solution to \( x^2 + 1 = 0 \)).
- A complex number can be written in two forms:
  - \( X = A + jB \)
    - \( A \) and \( B \) are real numbers
  - \( X = R \ e^{j\phi} \)
    - \( R = (A^2 + B^2)^{1/2} \) and \( \tan\phi = B/A \) (remember \( e^{j\phi} = \cos\phi + j \sin\phi \))
- Define the complex conjugate of \( X \) as:
  - \( X^* = A - jB \) or \( X^* = R \ e^{-j\phi} \)
- The magnitude of \( X \) can be found from:
  - \( |X| = (XX^*)^{1/2} = (X^* X)^{1/2} = (A^2 + B^2)^{1/2} \)
- Suppose we have 2 complex numbers, \( X \) and \( Y \) with phases \( \alpha \) and \( \beta \) respectively,
  - \( Z = \frac{X}{Y} = \frac{|X|e^{j\alpha}}{|Y|e^{j\beta}} = \frac{|X|}{|Y|} e^{j(\alpha-\beta)} \)
    - magnitude of \( Z \): \( |X|/|Y| \)
    - phase of \( Z \): \( \alpha - \beta \)

So why is this useful?
Consider the case of the capacitor and AC voltage:

\[ V(t) = V_0 \cos \omega t \]

\[ = \text{Real} \left( V_0 e^{j\omega t} \right) \]

\[ Q = CV \]

\[ I(t) = C \frac{dV}{dt} \]

\[ = -C \omega V_0 \sin \omega t \]

\[ = \text{Real} \left( j\omega CV_0 e^{j\omega t} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j\omega t}}{1/j\omega C} \right) \]

\[ = \text{Real} \left( \frac{V}{X_C} \right) \]

- V and \( X_C \) are complex numbers
- We now have Ohm's law for capacitors using the capacitive reactance \( X_C \):

\[ X_C = \frac{1}{j\omega C} \]
We can make a similar case for the inductor:

\[ V = L \frac{dI}{dt} \]

\[ I(t) = \frac{1}{L} \int V \, dt \]

\[ = \frac{1}{L} \int V_0 \cos \omega t \, dt \]

\[ = \frac{V_0 \sin \omega t}{L\omega} \]

\[ = \text{Real} \left( \frac{V_0 e^{j\omega t}}{j\omega L} \right) \]

\[ = \text{Real} \left( \frac{V}{X_L} \right) \]

- V and \( X_L \) are complex numbers.

We now have Ohm’s law for inductors using the inductive reactance \( X_L \):

\[ X_L = j\omega L \]

- \( X_C \) and \( X_L \) act like frequency dependent resistors.

- They also have a *phase* associated with them due to their complex nature.

- \( X_L \Rightarrow 0 \) as \( \omega \Rightarrow 0 \) (short circuit, DC)

- \( X_L \Rightarrow \infty \) as \( \omega \Rightarrow \infty \) (open circuit)

- \( X_C \Rightarrow 0 \) as \( \omega \Rightarrow \infty \) (short circuit)

- \( X_C \Rightarrow \infty \) as \( \omega \Rightarrow 0 \) (open circuit, DC)
Back to the RC circuit.
- Allow voltages, currents, and charge to be complex:
  \[ V_{in} = V_0 \cos \omega t \]
  \[ = \text{Re} \left( V_0 e^{j\omega t} \right) \]
  \[ = \text{Re} (V_R + V_C) \]
- We can write an expression for the charge \((Q)\) taking into account the phase difference \((\phi)\) between applied voltage and the voltage across the capacitor \((V_C)\).
  \[ Q(t) = CV_C(t) \]
  \[ = Ae^{j(\omega t - \phi)} \]
  \[ \square\] \(Q\) and \(V_C\) are complex
  \[ \square\] \(A\) and \(C\) are real
- We can find the complex current by differentiating the above:
  \[ I(t) = \frac{dQ(t)}{dt} \]
  \[ = j\omega Ae^{j(\omega t - \phi)} \]
  \[ = j\omega Q(t) \]
  \[ = j\omega CV_C(t) \]
  \[ V_{in} = V_C + V_R \]
  \[ = V_C + IR \]
  \[ = V_C + j\omega CV_C R \]
\[ V_C = \frac{V_{in}}{1 + j\omega RC} \]
\[ = V_{in} \frac{1}{j\omega C} \frac{1}{R + \frac{1}{j\omega C}} \]
\[ = V_{in} \frac{X_C}{R + X_C} \]

- looks like a voltage divider equation!!!!!

- We can easily find the magnitude of \( V_C \):

\[ |V_C| = |V_{in}| \left| \frac{|X_C|}{R + X_C} \right| \]
\[ = V_0 \frac{1}{\omega C} \frac{1}{\sqrt{R^2 + (1/\omega C)^2}} \]
\[ = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \]

- same as the result on page 4.
Is this solution the same as what we had when we solved by brute force page 4?

\[ V_C = \text{Real} \left( \frac{V_{\text{in}}}{1 + j\omega RC} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j\omega t}}{1 + j\omega RC} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j\omega t}}{\sqrt{1 + (\omega RC)^2} e^{j\phi}} \right) \]

\[ = \text{Real} \left( \frac{V_0 e^{j(\omega t - \phi)}}{\sqrt{1 + (\omega RC)^2}} \right) \]

\[ = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1 + (\omega RC)^2}} \]

\[ \phi \text{ is given by } \tan\phi = \omega RC \]

\[ \text{YES the solutions are identical.} \]
We can now solve for the voltage across the resistor.

- Start with the voltage divider equation in complex form:

\[ V_R = \frac{V_{in}R}{R + X_C} \]

\[ |V_R| = \frac{|V_{in}|R}{|R + X_C|} \]

\[ = \frac{V_0R}{\sqrt{R^2 + (1/\omega C)^2}} \]

\[ = \frac{V_0\omega RC}{\sqrt{1 + (\omega RC)^2}} \]

This amplitude is the same as the brute force differential equation case!

In adding complex voltages, we must take into account the phase difference between them.

- the sum of the voltages at a given time satisfy:

\[ V_0^2 = |V_R|^2 + |V_C|^2 \]

\[ V_0 = |V_R| + |V_C| \]

**R-C Filters**

- Allow us to select (reject) wanted (unwanted) signals on the basis of their frequency structure.
- Allow us to change the phase of the voltage or current in a circuit.
- Define the gain (\(G\)) or transfer (\(H\)) function of a circuit:
  - \(G(j\omega) = H(j\omega) = V_{out}/V_{in}\) (\(j\omega\) is often denoted by \(s\)).
  - \(G\) is independent of time, but can depend on \(\omega, R, L, C\).
For an RC circuit we can define $G_R$ and $G_C$:

\[ G_R \equiv \frac{V_R}{V_{in}} = \frac{R}{R + X_C} = \frac{R}{R + 1/j \omega C} \]

\[ G_C \equiv \frac{V_C}{V_{in}} = \frac{X_C}{R + X_C} = \frac{1/j \omega C}{R + 1/j \omega C} \]

We can categorize the $G$'s as follows:

<table>
<thead>
<tr>
<th>Frequency</th>
<th>$G_R$</th>
<th>$G_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Frequencies</td>
<td>≈ 1, no phase shift</td>
<td>≈ $1/j \omega CR \approx 0$, phase shift</td>
</tr>
<tr>
<td></td>
<td><strong>high pass filter</strong></td>
<td></td>
</tr>
<tr>
<td>Low Frequencies</td>
<td>≈ $j \omega CR \approx 0$, phase shift</td>
<td>≈ 1, no phase shift</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>low pass filter</strong></td>
</tr>
</tbody>
</table>

- **Decibels and Bode Plots:**
  - Decibel (dB) describes voltage or power gain:
    \[ \text{dB} = 20 \log(\frac{V_{out}}{V_{in}}) = 10 \log(\frac{P_{out}}{P_{in}}) \]
  - **Bode Plot** is a log-log plot with dB on the y axis and log($\omega$) or log($f$) on the x axis.
3 dB point or 3 dB frequency:
- also called break frequency, corner frequency, 1/2 power point
- At the 3 dB point:
  \[
  \frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}} \quad \text{since} \quad 3 = 20 \log \left( \frac{V_{out}}{V_{in}} \right)
  \]
  \[
  \frac{P_{out}}{P_{in}} = \frac{1}{2} \quad \text{since} \quad 3 = 10 \log \left( \frac{P_{out}}{P_{in}} \right)
  \]
- \( \omega RC = 1 \) for high or low pass filter
Phase vs frequency for capacitor

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