Lecture 4: R-L-C Circuits and Resonant Circuits

RLC series circuit:

- What's $V_R$?
  - Simplest way to solve for $V$ is to use voltage divider equation in complex notation:
    
    $$V_R = \frac{V_{in}R}{R + X_C + X_L}$$
    
    
    $$V_R = \frac{V_{in}R}{R + \frac{1}{j\omega C} + j\omega L}$$
    
    - Using complex notation for the apply voltage $V_{in} = V_0 \cos \omega t = \text{Real}(V_0 e^{j\omega t})$:
      
      $$V_R = \frac{V_0 e^{j\omega t}R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$
      
      - We are interested in the both the magnitude of $V_R$ and its phase with respect to $V_{in}$.
      - First the magnitude:
        
        $$|V_R| = \frac{|V_0 e^{j\omega t}|R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$
        
        $$|V_R| = \frac{V_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$
The phase of \( V_R \) with respect to \( V_{in} \) can be found by writing \( V_R \) in purely polar notation.

For the denominator we have:

\[
R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \exp\left\{j \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right\}
\]

Define the phase angle \( \phi \):

\[
\tan \phi = \frac{\text{Imaginary } X}{\text{Real } X} = \frac{\omega L - \frac{1}{\omega C}}{R}
\]

We can now write for \( V_R \) in complex form:

\[
V_R = \frac{V_o R e^{j\omega t}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{j\phi} = |V_R| e^{j(\omega t - \phi)}
\]

Depending on \( L, C, \) and \( \omega \), the phase angle can be positive or negative! In this example, if \( \omega L > 1/\omega C \), then \( V_R(t) \) lags \( V_{in}(t) \).

Finally, we can write down the solution for \( V \) by taking the real part of the above equation:

\[
V_R = \text{Real} \frac{V_o R e^{j(\omega t - \phi)}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_o R \cos(\omega t - \phi)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}
\]
$R = 100 \, \Omega$, $L = 0.1 \, \text{H}$, $C = 0.1 \, \mu\text{F}$

- $V_R \ll V_{in}$ at 100 Hz.
- $V_R$ and $V_{in}$ are not in phase at this frequency.
- The little wiggles on $V_R$ are real!
  - Transient solution (homogeneous solution) to the differential eq. describing the circuit.
  - After a few cycles this contribution to $V_R$ die out.
Bode plot of magnitude of $V_R/V_{in}$ vs. frequency
In general $V_C(t)$, $V_R(t)$, and $V_L(t)$ are all out of phase with the applied voltage.

$I(t)$ and $V_R(t)$ are in phase in a series RLC circuit.

The amplitude of $V_C$, $V_R$, and $V_L$ depend on $\omega$.

The table below summarizes the 3 cases with the following definitions:

$$Z = \left[ R^2 + (\omega L - 1/\omega C)^2 \right]^{1/2}$$

$$\tan \phi = (\omega L - 1/\omega C) / R$$

<table>
<thead>
<tr>
<th>Gain</th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_R/V_{in}$</td>
<td>$R/Z$</td>
<td>$-\phi$</td>
</tr>
<tr>
<td>$V_L/V_{in}$</td>
<td>$\omega L/Z$</td>
<td>$\pi/2 - \phi$</td>
</tr>
<tr>
<td>$V_C/V_{in}$</td>
<td>$1/\omega CZ$</td>
<td>$-\pi/2 - \phi$</td>
</tr>
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</table>

RLC circuits are resonant circuits

- energy in the system "resonates" between the inductor and capacitor
- "ideal" capacitors and inductors do not dissipate energy
- resistors dissipate energy i.e. resistors do not store energy
Resonant Frequency:
- At the resonant frequency the imaginary part of the impedance vanishes.
- For the series RLC circuit the impedance ($Z$) is:
  \[ Z = R + X_L + X_C = R + j(\omega L - 1/\omega C) \]
  \[ |Z| = \left[R^2 + (\omega L - 1/\omega C)^2\right]^{1/2} \]
- At resonance (series, parallel etc):
  \[ \omega L = 1/\omega C \]
  \[ \omega_R = \frac{1}{\sqrt{LC}} \]
- At the resonant frequency the following are true for a series RLC circuit:
  - $|V_R|$ is maximum (ideally $= V_{in}$)
  - $\phi = 0$
  - $\frac{|V_C|}{|V_{in}|} = \frac{|V_L|}{|V_{in}|} = \frac{\sqrt{L}}{R\sqrt{C}}$ ($V_C$ or $V_L$ can be $> V_{in}$!)
  - The circuit acts like a narrow band pass filter.

There is an exact analogy between an RLC circuit and a harmonic oscillator (mass attached to spring):
- $m \frac{d^2x}{dt^2} + B \frac{dx}{dt} + kx = 0$ damped harmonic oscillator
- $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$ undriven RLC circuit
- $x \Leftrightarrow q$ (electric charge), $L \Leftrightarrow m$, $k \Leftrightarrow 1/C$
- $B$ (coefficient of damping) $\Leftrightarrow R$
- **Q** (quality factor) of a circuit: determines how well the RLC circuit stores energy
  - $Q = 2\pi \frac{\text{max energy stored}}{\text{energy lost}}$ per cycle
  - $Q$ is related to sharpness of the resonance peak:
The maximum energy stored in the inductor is $LI_{MAX}^2/2$ with $I = I_{MAX}$.

- no energy is stored in the capacitor at this instant because $I$ and $V_C$ are $90^\circ$ out of phase.
- The energy lost in one cycle:

$$\text{power} \times (\text{time for cycle}) = I_{RMS}^2 R \times \frac{2\pi}{\omega_R} = \frac{1}{2} I_{MAX}^2 R \times \frac{2\pi}{\omega_R}$$

$$Q = \frac{2\pi \left( \frac{LI_{MAX}^2}{2} \right)}{2\pi \left( \frac{RI_{MAX}^2}{2} \right)} = \frac{\omega_R L}{R}$$

- There is another popular, equivalent expression for $Q$

$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

- $\omega_U$ ($\omega_L$) is the upper (lower) 3 dB frequency of the resonance curve.
  - $Q$ is related to sharpness of the resonance peak.
- Will skip the derivation here as it involves a bit of algebra.
  - two crucial points of the derivation:

$$\frac{V_R}{V_{in}} = \frac{1}{\sqrt{1 + Q^2 \left( \frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right)^2}}$$

$$Q \left( \frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right) = \pm 1 \quad \text{at the upper and lower 3 dB points}$$
- $Q$ can be measured from the shape of the resonance curve
  - one does not need to know $R$, $L$, or $C$ to find $Q$!

$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

- Example: Audio filter (band pass filter)
  - Audio filter is matched to the frequency range of the ear (20-20,000 Hz).
Let's design an audio filter using low and high pass RC circuits.

![Diagram of RC circuits](image)

- Ideally, the frequency response is flat over 20-20,000 Hz, and rolls off sharply at frequencies below 20 Hz and above 20,000 Hz.
  - Set 3 dB points as follows:
    - lower 3 dB point: $20 \text{ Hz} = \frac{1}{2 \pi R_1 C_1}$
    - upper 3 dB point: $2 \times 10^4 \text{ Hz} = \frac{1}{2 \pi R_2 C_2}$
  - If we put these two filters together we don't want the 2nd stage to affect the 1st stage.
    - can accomplish this by making the impedance of the 2nd ($Z_2$) stage much larger than $R_1$.
  - Remember $R_1$ is in parallel with $Z_2$.
    
    \[
    Z_1 = R_1 + 1/j\omega C_1 \\
    Z_2 = R_2 + 1/j\omega C_2
    \]
  - In order to insure that the second stage does not "load" down the first stage we need:
    
    \[R_2 >> R_1\] since at high frequencies $Z_2 \Rightarrow R_2$
  - We can now pick and calculate values for the $R$'s and $C$'s in the problem.
    - Let $C_1 = 1 \mu F \Rightarrow R_1 = 1/(20\text{ Hz} \cdot 2\pi C_1) = 8 \text{ k}\Omega$
    - Let $R_2 > 100R_1 \Rightarrow R_2 = 1 \text{ M}\Omega$, and $C_2 = 1/(2 \times 10^4 \text{ Hz} \cdot 2\pi R_2) = 8 \text{ pf}$
    - \[R_1 = 8 \text{ k}\Omega, C_1 = 1 \mu F\]
    \[R_2 = 1 \text{ M}\Omega, C_2 = 8 \text{ pf}\]
Exact derivation for above filter:

- In the above circuit we treated the two RC filters as independent.
- Why did this work?
- We want to calculate the gain ($|V_{out}/V_{in}|$) of the following circuit:

\[ V_{out} = V_a X_2 / (X_2 + R_2) \]
\[ V_a = V_{in} Z_1 / Z_T \]

- $Z_T$ is the total impedance of the circuit as seen from the input.
- $Z_1$ is the parallel impedance of $R_1$ and $R_2$, in series with $C_2$.

\[ Z_1 = \frac{R_1 (R_2 + X_2)}{R_1 + R_2 + X_2} \]
\[ Z_T = X_1 + Z_1 \]

Finally we can solve for the gain $G = |V_{out}/V_{in}|$:

\[ \frac{V_{out}}{V_{in}} = \frac{R_1 X_2}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)} \]
We can relate this to our previous result by rewriting the above as:

\[
\frac{V_{out}}{V_{in}} = \frac{R_1 \frac{X_2}{R_2 + X_2}}{X_1 \left( \frac{R_1}{R_2 + X_2} + 1 \right) + R_1}
\]

If we now remember the approximation \((R_1 << R_2 + X_2)\) made on the previous page to insure that the second stage did not load down the first then we get the following:

\[
\frac{V_{out}}{V_{in}} = \frac{R_1}{R_1 + X_1} \frac{X_2}{R_2 + X_2}
\]

The gain of the circuit looks like the product of two filters, one high pass and one low pass!

If we calculate the gain of this circuit in dB, the total gain is the sum of the gain of each piece:

Gain in dB = \(20 \log \left( \frac{V_{out}}{V_{in}} \right)\)

\[
= 20 \log \left( \frac{R_1}{R_1 + X_1} \right) + 20 \log \left( \frac{X_2}{R_2 + X_2} \right)
\]

\(\checkmark\) The gain of successive filters measured in dB's add!
Another Example: Calculate $|I|$ and the phase angle between $V_{in}$ and $I$ for the following circuit:

First calculate $|I|$.
- The total current out of the input source ($I$) is related to $V_{in}$ and the total impedance ($Z_T$) of the circuit by Ohm’s law:
  $$I = \frac{V_{in}}{Z_T}$$
- The total impedance of the circuit is given by the parallel impedance of the two branches:
  $$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2}$$
  $$Z_1 = R_1 + X_1$$
  $$Z_2 = R_2 + X_2$$
- Putting in numerical values for the $R$'s and $X$'s we have:
  $$Z_1 = 20 + j37.7 \ \Omega$$
  $$Z_2 = 10 - j53.1 \ \Omega$$
  $$Z_T = 67.4 + j11.8 \ \Omega$$
- We can now find the magnitude of the current:
  $$|I| = \frac{|V_{in}|}{|Z_T|}$$
  $$= \frac{230 \ \text{V}}{68.4 \ \Omega}$$
  $$= 3.36 \ \text{A}$$

This is RMS value since $|V_{in}|$ is given as RMS
Calculate the phase angle between $V_{in}$ and $I$:

- It’s easiest to solve this by writing $V$ and $Z$ in polar form:
  
  $V_{in} = (230)e^{j\omega t}$
  
  $Z_T = (68.4)e^{j\phi}$
  
  \[
  \tan \phi = \frac{\text{Im} Z_T}{\text{Re} Z_T}
  \]
  
  $\phi = \frac{11.8}{67.4} \approx 9.9^0$

- Finally we can write for the current:
  
  $I = \frac{230}{68.4}e^{j(\omega t - \phi)}$

- Taking the real part of $I$:
  
  $I = 3.36 \cos(\omega t - 9.9^0) \text{ A}$

- The current lags the voltage by $9.9^0$. 