Physics 829
Dr. Herbst

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Answers To Final Examination

① $A_z = A_0 e^{i(ky-\omega t)} + A_0^* e^{-i(ky-\omega t)}$

$$W' = \frac{1}{2m} q^2 A_z^2 = \frac{q^2}{2m}\left(\frac{B}{2ik} e^{i(ky-\omega t)} - \frac{B}{2ik} e^{-i(ky-\omega t)}\right)^2$$

$$W' = -\frac{q^2 B^2}{2mk^2}\,\frac{1}{4}\left\{ \underbrace{e^{2i(ky-\omega t)}}_{\text{absorption}} + \underbrace{e^{-2i(ky-\omega t)}}_{\text{emission}} + 2 \right\}$$

no transition possible since no spatial part

Rotating wave approx:
use 1ST Term only for absorption

$$W' = -\frac{q^2 B^2}{8mk^2} e^{2iky} e^{-2i\omega t}$$

Let $\omega' = 2\omega$

$$e^{2iky} \simeq 1 + 2iky$$

causes no transitions

$$W' = \left(-\frac{q^2 B^2}{8mk^2} 2iky\right) e^{-2i\omega t}$$

$$W_{fi} = \langle 2p | \frac{-i q^2 B^2 y}{4mk} | 1s \rangle = -\frac{i q^2 B^2}{4mk} \langle 2p | y | 1s \rangle$$

$$|W_{fi}|^2 = \frac{q^4 B^4}{16 m^2 k^2} |\langle 2p | y | 1s \rangle|^2 = \frac{q^2 B^4}{16 m^2 k^2} |\langle 2p | \mu_y | 1s \rangle|^2$$

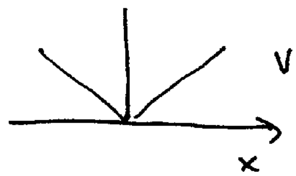
$$P_{fi} = \frac{|W_{fi}|^2}{\hbar^2} \frac{\sin^2 (\omega' - \omega_{fi}) t/2}{[(\omega' - \omega_{fi})/2]^2} \qquad \omega' = 2\omega$$

no "4" needed since $e^{-i\omega' t}$ used not $\sin(\omega' t)$

$$P_{2p\leftarrow 1s} = \frac{q^2 B^4}{16 \hbar^2 m^2 k^2} |\langle 2p | \mu_y | 1s \rangle|^2 \frac{\sin^2 [(2\omega - \omega_{fi}) t/2]}{[(2\omega - \omega_{fi})/2]^2}$$

resonance condition: $2\omega = \omega_{fi}$ "two-photon" transition

(2)



$V = k|x| \qquad \psi = e^{-\alpha x^2}$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 2\int_0^{\infty} e^{-2\alpha x^2} dx = \sqrt{\frac{\pi}{2\alpha}}$$

$$\langle \mathcal{H} \rangle = \sqrt{\frac{2\alpha}{\pi}} \left\{ \langle T \rangle + \langle V \rangle \right\}$$

$$\langle V \rangle = 2k\int_0^{\infty} e^{-2\alpha x^2} x\, dx = (2k/-4\alpha)\, e^{-2\alpha x^2} \Big|_0^{\infty} = k/2\alpha$$

$$\frac{d}{dx} e^{-\alpha x^2} = -2\alpha x\, e^{-\alpha x^2} \qquad \frac{d^2}{dx^2} e^{-\alpha x^2} = (4\alpha^2 x^2 - 2\alpha)\, e^{-\alpha x^2}$$

$$\langle T \rangle = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-2\alpha x^2} (4\alpha^2 x^2 - 2\alpha)\, dx$$

$$= -\frac{\hbar^2}{2m} \left\{ \frac{2\cdot 4\alpha^2\, \Gamma(3/2)}{2\,(2\alpha)^{3/2}} - 2\alpha\sqrt{\frac{\pi}{2\alpha}} \right\} ; \quad \Gamma(3/2) = \frac{1}{2}\pi^{1/2}$$

$$\langle T \rangle = \frac{\hbar^2}{2m} \left\{ 2\alpha\sqrt{\frac{\pi}{2\alpha}} - \frac{2\alpha^2 \pi^{1/2}}{(2\alpha)^{3/2}} \right\}$$

$$\langle \mathcal{H} \rangle = \frac{\hbar^2}{2m} \left\{ 2\alpha - \alpha \right\} + \frac{k}{\sqrt{2\pi}}\, \alpha^{-1/2}$$

$$\langle \mathcal{H} \rangle = \frac{\hbar^2 \alpha}{2m} + \frac{k}{\sqrt{2\pi}}\, \alpha^{-1/2}$$

$$\frac{d\langle \mathcal{H} \rangle}{d\alpha} = \frac{\hbar^2}{2m} - \frac{1}{2}\frac{k}{\sqrt{2\pi}}\, \alpha^{-3/2} = 0 \Rightarrow \alpha^{3/2} = \frac{k}{\sqrt{2\pi}}\, \frac{m}{\hbar^2}$$

$$\alpha = \left(\frac{mk}{\sqrt{2\pi}\, \hbar^2} \right)^{2/3}$$

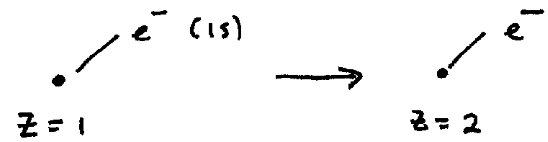
$$\langle \mathcal{H} \rangle = \frac{\hbar^2}{2m} \left(\frac{mk}{\sqrt{2\pi}\, \hbar^2} \right)^{2/3} + \frac{k}{\sqrt{2\pi}} \left(\frac{\sqrt{2\pi}\, \hbar^2}{mk} \right)^{1/3}$$

$$\langle \mathcal{H} \rangle = \left(\frac{\hbar^2 k^2}{m} \right)^{1/3} \left[\frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \right)^{2/3} + \frac{1}{\sqrt{2\pi}} \left(\sqrt{2\pi} \right)^{1/3} \right]$$

$$\langle \mathcal{H} \rangle = \left(\frac{\hbar^2 k^2}{m} \right)^{1/3} \left[\frac{3}{2} \left(\frac{1}{\sqrt{2\pi}} \right)^{2/3} \right] = 0.8129 \left(\frac{\hbar^2 k^2}{m} \right)^{1/3}$$

higher than actual E_0 by $< 1\%$.

(3)



sudden approx.

a) $\text{Prob} = |\langle 1s(Z=2) | 1s(Z=1) \rangle|^2$

$$\langle 1s(Z=2) | 1s(Z=1) \rangle = \frac{\sqrt{1^3}\,\sqrt{2^3}}{\pi a_0^3} \int_{r=0}^{\infty} r^2 e^{-r/a_0} e^{-2r/a_0} dr\, 4\pi$$

$$\langle 1s(Z=2) | 1s(Z=1) \rangle = \frac{4 \cdot 2\sqrt{2}}{a_0^3} \underbrace{\int_{r=0}^{\infty} r^2 e^{-3r/a_0} dr}_{2!\,/\,(3/a_0)^3}$$

$$\langle 1s(Z=2) | 1s(Z=1) \rangle = 16\sqrt{2}/27 \qquad \text{Prob} = \left(\frac{16}{27}\sqrt{2}\right)^2 = 0.702$$

b) Selection Rules

$$\psi_{1s}(Z=1) = R_{10}(r)\, Y_0^0(\theta,\varphi)$$

$$\psi'_{n\ell m}(Z=2) = R'_{n\ell}(r)\, Y_\ell^m(\theta,\varphi)$$

$$\langle n\ell m(Z=2) | 1s(Z=1) \rangle \propto \underbrace{\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_\ell^{m*}\, Y_0^0\, d\Omega}_{0 \text{ unless } \ell=0\ m=0}$$

$\Delta\ell = 0 \quad \Delta m = 0 \quad \Longleftarrow$

$\therefore$ only ns states can be reached.

(4)

—— 2s —— 2d $3.5\hbar\omega$

—— 1p $2.5\hbar\omega$

—— 0s $1.5\hbar\omega$

$|\psi\rangle = |\psi_A\rangle$

a) i) lowest state $0s^2$ $\quad E = 3\hbar\omega$ $\quad \underline{\uparrow\downarrow}$ 0s

$$|\psi_A\rangle = |1{:}0s\rangle |2{:}0s\rangle\, |S=0\ M_S=0\rangle \qquad {}^1S \qquad g=1$$

ii) 1st excited configuration $0s^1\, 1p^1$ $\quad E = 4\hbar\omega$

$L=1$

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left\{ |1:0s\rangle|2:1p\rangle - |1:1p\rangle|2:0s\rangle \right\} |S=1\, M_S\rangle$$
"^3P"

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left\{ |1:0s\rangle|2:1p\rangle + |1:1p\rangle|2:0s\rangle \right\} |S=0\, M_S=0\rangle$$
"^1P"

$$g = 9+3 = 12$$

(iii) 2nd excited configuration $\quad 0s^1 2s^1 \quad 0s^1 2d^1 \quad 1p^2$

$E = 5\hbar\omega$

$0s^1 2s^1$: $|\psi_A\rangle = \frac{1}{\sqrt{2}}\left\{ |1:0s\rangle|2:2s\rangle - |1:2s\rangle|2:0s\rangle \right\} |S=1\, M_S\rangle$

^3S

^1S $\quad |\psi_A\rangle = \frac{1}{\sqrt{2}}\left\{ |1:0s\rangle|2:2s\rangle + |1:2s\rangle|2:0s\rangle \right\} |S=0,0\rangle$

$0s^1 2d^1$

$L=2$ $\quad {}^3D \quad |\psi_A\rangle = \frac{1}{\sqrt{2}}\left\{ |1:0s\rangle|2:2d\rangle - |1:2d\rangle|2:0s\rangle \right\} |S=1\, M_S\rangle$

$^1D \quad |\psi_A\rangle = \frac{1}{\sqrt{2}}\left\{ |1:0s\rangle|2:2d\rangle + |1:2d\rangle|2:0s\rangle \right\} |S=0,0\rangle$

$1p^2 \quad \ell_1 = 1 \quad \ell_2 = 1 \quad L = 2, 1, 0$

sym $\quad$ anti $\quad$ symm

$$|\psi_A\rangle = |L=2, M_L\rangle\, |S=0, M_S=0\rangle \quad {}^1D$$

$$|\psi_A\rangle = |L=1, M_L\rangle\, |S=1, M_S\rangle \quad {}^3P$$

$$|\psi_A\rangle = |L=0, M_L=0\rangle\, |S=0\, M_S=0\rangle \quad {}^1S$$

Total degeneracy $= 3+1 + 15+5 + 5+9+1 = 39$

——— $0s^1 2s^1$ $\;{}^3S\;{}^1S$ ——— $0s^1 2d^1$ $\;{}^1D\;{}^3D$ ——— $1p^2$ $\;{}^1D\;{}^3P\;{}^1S$ $\quad g = 39$

——— $0s^1 1p^1$ $\;{}^3P\;{}^1P$ $\quad g = 12$

——— $0s^2$ $\;{}^1S$ $\quad g = 1$

b) $0s^2\, 1p^1 \quad {}^2P \quad g = 6 \quad E = 5.5\,\hbar\omega$

$$|\psi_A\rangle = \frac{1}{\sqrt{3!}} \begin{vmatrix} |1: 0s+\rangle & |1: 0s-\rangle & |1: 1p_x \pm\rangle \\ |2: 0s+\rangle & |2: 0s-\rangle & |2: 1p_x \pm\rangle \\ |3: 0s+\rangle & |3: 0s-\rangle & |3: 1p_x \pm\rangle \end{vmatrix}$$

$x(m_\ell) = 1, 0, -1$