Physics 829

Spring 2000
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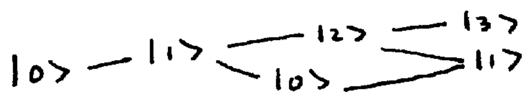
Answers To Midterm Examination

(1) $W = ax^3$ - odd $\therefore \langle 0|ax^3|0\rangle = 0$

$$E_0 = \frac{\hbar\omega}{2} + \sum_n \frac{|\langle 0|W|n\rangle|^2}{E_0^0 - E_n^0} = \frac{\hbar\omega}{2} - \frac{a^2}{\hbar\omega}\sum_n \frac{1}{n}|\langle 0|x^3|n\rangle|^2$$



$$\langle 0|x^3|3\rangle = X_{01}X_{12}X_{23} = (\hbar/2\mu\omega)^{3/2}\sqrt{1\cdot 2\cdot 3}$$

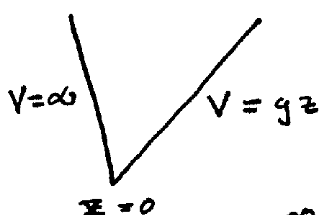
$$\langle 0|x^3|1\rangle = X_{01}X_{12}X_{21} + X_{01}X_{10}X_{01} = \left(\frac{\hbar}{2\mu\omega}\right)^{3/2}\left\{\sqrt{4} + \sqrt{1}\right\}$$

$$= (\hbar/2\mu\omega)^{3/2}(2+1)$$

$$E_0 = \frac{\hbar\omega}{2} - \frac{a^2}{\hbar\omega}\left\{\frac{1}{1}\left(\frac{\hbar}{2\mu\omega}\right)^3 9 + \frac{1}{3}\left(\frac{\hbar}{2\mu\omega}\right)^3 6\right\}$$

$$E_0 = \frac{\hbar\omega}{2} - \frac{a^2\hbar^2\cdot 11}{8\mu^3\omega^4}$$

(2)



Trial $\psi(z) = z e^{-az}$

$$\langle\psi|\psi\rangle = \int_0^\infty z^2 e^{-2az}\,dz = \frac{2!}{(2a)^3}$$

$$\langle\psi|V|\psi\rangle = g\int_0^\infty z^3 e^{-2az}\,dz = g\,3!/(2a)^4$$

$$\frac{d^2\psi}{dz^2} = \frac{d}{dz}\left(e^{-az} - aze^{-az}\right) = e^{-az}\left(-2a + a^2 z\right)$$

$$\langle\psi|T|\psi\rangle = -\hbar^2/2m\,\langle\psi|\frac{d^2\psi}{dz^2}\rangle = -\frac{\hbar^2}{2m}\int_0^\infty e^{-2az}(-2az + a^2z^2)\,dz$$

$$\langle\psi|T|\psi\rangle = -\frac{\hbar^2}{2m}\left\{(-2a)\,1!/(2a)^2 + a^2\,2!/(2a)^3\right\}$$

$$\langle\psi|T|\psi\rangle = -\frac{\hbar^2}{2m}\left\{-\frac{1}{2a} + \frac{1}{4a}\right\} = \hbar^2/8am$$

$$\langle\mathcal{H}\rangle = \frac{\hbar^2/8am + 3g/8a^4}{2/8a^3} = \frac{\hbar^2 a^2}{2m} + \frac{3g}{2a}$$

$$\frac{d\langle\mathcal{H}\rangle}{da} = 0 = \frac{\hbar^2}{m}a + \frac{3g}{2}\left(-a^{-2}\right) \Rightarrow \frac{\hbar^2}{m}a^3 = \frac{3}{2}g$$

$$a^3 = \frac{3mg}{2\hbar^2} \qquad a = \left(\frac{3mg}{2\hbar^2}\right)^{1/3}$$

$$\langle\mathcal{H}\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{3mg}{2\hbar^2}\right)^{2/3} + \frac{3g}{2}\left(\frac{2\hbar^2}{3mg}\right)^{1/3}$$

$$\langle\mathcal{H}\rangle_{min} = \frac{g^{2/3}\hbar^{2/3}}{m^{1/3}}\left[\frac{1}{2}\left(\frac{3}{2}\right)^{2/3} + \frac{3}{2}\left(\frac{2}{3}\right)^{1/3}\right]$$

$$= \left(\frac{g^2\hbar^2}{m}\right)^{1/3}\left[\left(\frac{3}{2}\right)^{5/3}\right] = 1.966\left(\frac{g^2\hbar^2}{m}\right)^{1/3}$$

<u>WKB</u>

$$\int_0^{z_T} k\,dz = \pi(n+1/2)\,; \quad k = \sqrt{2m(E-gz)}/\hbar$$

z_T defined by $E = gz$

$$\frac{\sqrt{2m}}{\hbar}\int_0^{z_T}(E-gz)^{1/2}\,dz = \pi(n+1/2)$$

$$\frac{\sqrt{2m}}{\hbar}\left.\frac{2(E-gz)^{3/2}}{3(-g)}\right|_0^{z_T} = \pi(n+1/2)$$

$$\frac{\sqrt{2m}}{\hbar}\frac{2}{3g}E_n^{3/2} = \pi(n+1/2)$$

$$E_n^{3/2} = \frac{3 g \hbar \pi (n+1/2)}{2\sqrt{2m}}$$

$$E_n = \left(\frac{3 g \hbar \pi (n+1/2)}{2^{3/2} m^{1/2}}\right)^{2/3}$$

$$E_0 = \left(\frac{g^2 \hbar^2}{m}\right)^{1/3} \left(\frac{3 \cdot 1/2 \cdot \pi}{2^{3/2}}\right)^{2/3}$$

$$E_0 = \left(\frac{g^2 \hbar^2}{m}\right)^{1/3} \underbrace{\frac{1}{2}\left(\frac{3\pi}{2}\right)^{2/3}}_{1.405}$$

——— VAR

——— WKB

(3) (a) $$P_{2p_0,1s}(\omega=0,\tau) = \frac{|W_{fi}|^2}{\hbar^2} \frac{\sin^2(\omega_{fi}\tau/2)}{(\omega_{fi}/2)^2}$$

$$W = e z E_z \qquad W_{fi} = e E_z \langle 2p_0|z|1s\rangle = 0.74 e a_0 E_z$$

$$P_{2p_0,1s}(\omega=0,\tau) = \frac{(0.74)^2 e^2 a_0^2 E_z^2}{\hbar^2} \frac{\sin^2(\omega_{fi}\tau/2)}{(\omega_{fi}/2)^2}$$

$$\omega_{fi} = \frac{E_{2p_0} - E_{1s}}{\hbar} = 3/8\, e^2/a_0\hbar$$

(b) $\omega = \omega_{fi}$ $$P_{2p_0,1s}(\omega\approx\omega_{fi},\tau) = \frac{|W_{fi}|^2}{4\hbar^2}\tau^2$$

$$P_{2p_0,1s}(\omega\approx\omega_{fi},\tau) = \frac{(0.74)^2 e^2 a_0^2 E_z^{02}}{4\hbar^2}\tau^2$$

(c) $$P = \frac{(0.74)^2 e^2 a_0^2 E_z^{02}}{4\hbar^2}\int_0^\infty g(\omega)\frac{\sin^2(\omega-\omega_{fi})\tau/2}{[(\omega-\omega_{fi})/2]^2}\,d\omega$$

$$I \cong \int_{\approx -\infty}^{\infty} \frac{\sin^2 (\omega - \omega_{fi})\,\tau/2}{\left(\frac{(\omega-\omega_{fi})}{2}\right)^2}\, d(\omega - \omega_{fi})$$

$$x = (\omega - \omega_{fi})\,\tau/2 \qquad dx = d(\omega - \omega_{fi})\,\tau/2$$

$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{(x/\tau)^2}\,\frac{2}{\tau}\,dx = \underbrace{2\tau\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx}_{2\tau\pi}$$

$$P_{if} = \frac{(0.74)^2 e^2 a_0^2 E_z^{0\,2}}{4\hbar^2}\, g(\omega_{fi})\, 2\tau\pi$$

$$= \pi \frac{(0.74)^2 e^2 a_0^2}{2\hbar^2} E_z^{0\,2}\, g(\omega_{fi})\,\tau$$

or $$\Gamma_{if} = \frac{\pi}{2\hbar^2} (0.74)^2 e^2 a_0^2 E_z^{0\,2}\, g(\omega_{fi})$$

same as long-time result!