Table 1. Harmonic Polynomials and Spherical Harmonics.

$$\mathcal{Y}_{lm}(\mathbf{r}) = r^l Y_{lm}(\theta, \varphi)$$

l, m	$\mathcal{Y}_{lm}(\mathbf{r})$	$Y_{lm}(\theta, \varphi)$
0 0	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$
1 0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\, z$	$\frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$
1 ±1	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\,(x \pm iy)$	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{\pm i\varphi}$
2 0	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\,(2z^2 - x^2 - y^2)$	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\,(2\cos^2\theta - \sin^2\theta)$
2 ±1	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\, z(x \pm iy)$	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cos\theta \sin\theta e^{\pm i\varphi}$
2 ±2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\,(x \pm iy)^2$	$\frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{\pm 2i\varphi}$
3 0	$\frac{1}{4}\sqrt{\frac{7}{\pi}}\,(2z^2 - 3x^2 - 3y^2)z$	$\frac{1}{4}\sqrt{\frac{7}{\pi}}\,(2\cos^3\theta - 3\cos\theta\sin^2\theta)$
3 ±1	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}\,(4z^2 - x^2 - y^2)(x \pm iy)$	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}\,(4\cos^2\theta\sin\theta - \sin^3\theta)e^{\pm i\varphi}$
3 ±2	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}\, z(x \pm iy)^2$	$\frac{1}{4}\sqrt{\frac{105}{2\pi}} \cos\theta \sin^2\theta e^{\pm 2i\varphi}$
3 ±3	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}}\,(x \pm iy)^3$	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}} \sin^3\theta e^{\pm 3i\varphi}$

Irreducible tensors containing in addition the components of some other vector $\mathbf{r}'$ may be constructed by polarization of the harmonics with the operator

$$\mathbf{r}' \cdot \nabla \equiv x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z}$$

Cf. Rose (1954).

Table 2.

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$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{1}{2}J} \left[\frac{(j_1 + j_2 - j_3)!(j_1 + j_3 - j_2)!(j_2 + j_3 - j_1)!}{(j_1 + j_2 + j_3 + 1)!} \right]^{\frac{1}{2}} \frac{(\frac{1}{2}J)!}{(\frac{1}{2}J - j_1)!(\frac{1}{2}J - j_2)!(\frac{1}{2}J - j_3)!}$$

if J is even.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{if} \quad J \text{ is odd} \quad \text{where} \quad J = j_1 + j_2 + j_3$$

$$\begin{pmatrix} J+\frac{1}{2} & J & \frac{1}{2} \\ M & -M-\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (-1)^{J-M-\frac{1}{2}} \left[\frac{J - M + \frac{1}{2}}{(2J + 2)(2J + 1)} \right]^{\frac{1}{2}} \qquad (J+\tfrac{1}{2}, J, \tfrac{1}{2})$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M-1} \left[\frac{(J - M)(J - M + 1)}{(2J + 3)(2J + 2)(2J + 1)} \right]^{\frac{1}{2}}$$

$$(J+1, J, 1)$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M-1} \left[\frac{(J + M + 1)(J - M + 1)\cdot 2}{(2J + 3)(2J + 2)(2J + 1)} \right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M} \left[\frac{(J - M)(J + M + 1)\cdot 2}{(2J + 2)(2J + 1)(2J)} \right]^{\frac{1}{2}}$$

$$(J, J, 1)$$

$$\begin{pmatrix} J & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M} \frac{M}{[(2J + 1)(J + 1)J]^{\frac{1}{2}}}$$

Table 2 (continued)

$\begin{pmatrix} J+\frac{3}{2} & J & \frac{3}{2} \\ M & -M-\frac{3}{2} & \frac{3}{2} \end{pmatrix}$	$(-1)^{J-M+\frac{1}{2}}\left[\frac{(J-M-\frac{1}{2})(J-M+\frac{1}{2})(J-M+\frac{3}{2})}{(2J+4)(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$	$(J+\frac{3}{2}, J, \frac{3}{2})$
$\begin{pmatrix} J+\frac{3}{2} & J & \frac{3}{2} \\ M & -M-\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$(-1)^{J-M+\frac{1}{2}}\left[\frac{3(J-M+\frac{1}{2})(J-M+\frac{3}{2})(J+M+\frac{3}{2})}{(2J+4)(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$	
$\begin{pmatrix} J+\frac{1}{2} & J & \frac{3}{2} \\ M & -M-\frac{3}{2} & \frac{3}{2} \end{pmatrix}$	$(-1)^{J-M-\frac{1}{2}}\left[\frac{3(J-M-\frac{1}{2})(J-M+\frac{1}{2})(J+M+\frac{3}{2})}{(2J+3)(2J+2)(2J+1)2J}\right]^{\frac{1}{2}}$	$(J+\frac{1}{2}, J, \frac{3}{2})$
$\begin{pmatrix} J+\frac{1}{2} & J & \frac{3}{2} \\ M & -M-\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$(-1)^{J-M-\frac{1}{2}}\left[\frac{J-M+\frac{1}{2}}{(2J+3)(2J+2)(2J+1)2J}\right]^{\frac{1}{2}}(J+3M+\frac{3}{2})$	
$\begin{pmatrix} J+2 & J & 2 \\ M & -M-2 & 2 \end{pmatrix}$	$(-1)^{J-M}\left[\frac{(J-M-1)(J-M)(J-M+1)(J-M+2)}{(2J+5)(2J+4)(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$	
$\begin{pmatrix} J+2 & J & 2 \\ M & -M-1 & 1 \end{pmatrix}$	$2(-1)^{J-M}\left[\frac{(J+M+2)(J-M+2)(J-M+1)(J-M)}{(2J+5)(2J+4)(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$	$(J+2, J, 2)$
$\begin{pmatrix} J+2 & J & 2 \\ M & -M & 0 \end{pmatrix}$	$(-1)^{J-M}\left[\frac{6(J+M+2)(J+M+1)(J-M+2)(J-M+1)}{(2J+5)(2J+4)(2J+3)(2J+2)(2J+1)}\right]^{\frac{1}{2}}$	

$$\begin{pmatrix} J+1 & J & 2 \\ M & -M-2 & 2 \end{pmatrix} \quad 2(-1)^{J-M+1}\left[\frac{(J-M-1)(J-M)(J-M+1)(J+M+2)}{(2J+4)(2J+3)(2J+2)(2J+1)2J}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J+1 & J & 2 \\ M & -M-1 & 1 \end{pmatrix} \quad (-1)^{J-M+1}2(J+2M+2)\left[\frac{(J-M+1)(J-M)}{(2J+4)(2J+3)(2J+2)(2J+1)2J}\right]^{\frac{1}{2}} \qquad (J+1, J, 2)$$

$$\begin{pmatrix} J+1 & J & 2 \\ M & -M & 0 \end{pmatrix} \quad (-1)^{J-M+1}2M\left[\frac{6(J+M+1)(J-M+1)}{(2J+4)(2J+3)(2J+2)(2J+1)2J}\right]^{\frac{1}{2}}$$

$$-\begin{pmatrix} J & J & 2 \\ M & -M-2 & 2 \end{pmatrix} \quad (-1)^{J-M}\left[\frac{6(J-M-1)(J-M)(J+M+1)(J+M+2)}{(2J+3)(2J+2)(2J+1)(2J)(2J-1)}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J & J & 2 \\ M & -M-1 & 1 \end{pmatrix} \quad (-1)^{J-M}(1+2M)\left[\frac{6(J+M+1)(J-M)}{(2J+3)(2J+2)(2J+1)(2J)(2J-1)}\right]^{\frac{1}{2}} \qquad (J, J, 2)$$

$$\begin{pmatrix} J & J & 2 \\ M & -M & 0 \end{pmatrix} \quad (-1)^{J-M}\frac{2[3M^2 - J(J+1)]}{[(2J+3)(2J+2)(2J+1)(2J)(2J-1)]^{\frac{1}{2}}}$$