Complete Motion

$$\eta_j = a_{ji} \zeta_i = \frac{a_{ji} Q_{oi}}{\omega_i^2 - \omega^2} \cos(\omega t + \delta_i)$$

each normal oscillation at frequency of driving force

summation over modes

Extent of each normal mode excitation: (B_i)

(i) amplitude Q_{oi} depends on component of force in the direction of normal mode vibration

(ii) closeness of driving frequency to free mode frequency

$\omega = \omega_i$ "resonance" ∞ contribution

Theory is for infinitesimal amplitudes only, not large ones.

- discussion unrealistic in the absence of dissipative or frictional forces.

START →

Dissipation

- in many instances, dissipative or frictional forces are proportional to velocities. If so, can be derived from a dissipation function F:

(1.5) $$F = \frac{1}{2} \sum_i k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2$$

$$F_{fx_i} = -\frac{\partial F}{\partial v_{ix}} = -k_x v_{ix}$$

Now, we generalize to

$$F = \frac{1}{2} F_{ij} \dot{\eta}_i \dot{\eta}_j \quad \text{a quadratic form}$$

$$F_{ij} = F_{ji}$$

$$Q_k = -\frac{\partial F}{\partial \dot\eta_k} = -\left[\frac{1}{2} F_{ik}\,\dot\eta_i + \frac{1}{2} F_{kj}\,\dot\eta_j\right]$$
$$= -F_{kj}\,\dot\eta_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\eta_i} - \frac{\partial L}{\partial \eta_i} = Q_i \qquad L = \frac{1}{2} T_{ij}\dot\eta_i\dot\eta_j - \frac{1}{2} V_{ij}\eta_i\eta_j$$

$$T_{ij}\ddot\eta_j + V_{ij}\eta_j + F_{ij}\dot\eta_j = 0$$

To find normal coordinates, must simultaneously diagonalize T, V, F matrices. Not generally possible.

But, if $F \propto T$ is possible:

in normal coordinates (*no summation*) $\quad \ddot\zeta_i + F_i\dot\zeta_i + \omega_i^2\zeta_i = 0$

$\uparrow$ non-negative coefficients of diagonalized matrix F_{ij} (otherwise blow-up, not dissipation)

$$\zeta_i = C_i e^{-i\omega_i' t}$$

$$-\omega_i'^2 - i\omega_i' F_i + \omega_i^2 = 0$$
$$\omega_i'^2 + i\omega_i' F_i - \omega_i^2 = 0$$
$$\omega_i' = \left[-iF_i \pm \sqrt{-F_i^2 + 4\omega_i^2}\,\right]/2$$
$$\omega_i' = \pm\underbrace{\sqrt{\omega_i^2 - F_i^2/4}}_{\omega_0^2} \quad \underbrace{-\,iF_i/2}$$

Choose + (physically meaningful)

$\Rightarrow$ exponential decay term

Underdamping

$\omega_i^2 > F_i^2/4$

$\omega_0^2 > 0$

$$\zeta_i = C_i e^{-i\omega_0 t} e^{-F_i t/2}$$

Critical Damping

$\omega_i^2 = F_i^2/4 \qquad \mathcal{J}_i = C_i\, e^{-F_i t/2}$

$\omega_0^2 = 0$

exponential decay

($A_i + B_i t$ more general)

Overdamping

$0 < \omega_i^2 < F_i^2/4 \qquad \mathcal{J}_i = C_i\, e^{-(F_i/2 \quad |\omega_0|)t}$

$-F_i^2/4 < \omega_0^2 < 0$

- standard meaning

Forcing + Damping

resonances occur in underdamped case, but messy!

Chapter 8 The Hamiltonian Equations of Motion

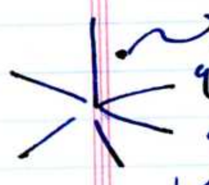
- alternative approach to classical mechanics for
- useful as basis for quantum mechanics and for numerical solutions (1st order not 2nd order)

- we will assume that the systems are holonomic and that the forces are monogenic (derived from a potential dependent on position only, or from velocity-dependent generalized potentials)

8.1 Legendre Transformations

Re: Lagrangian system $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ 2nd order

+ 2n initial values. Independent variables are the n generalized coordinates q_i, which vary in a configuration space.



→ Time derivatives not independent coordinates in a sense since no special equation for them.

Hamiltonian system 2n 1st order differential equations expressed in terms of 2n independent coordinates that vary in phase space. As shall be shown:

n q_i's and n conjugate momenta p_i's

where $p_i = \frac{\partial L}{\partial \dot{q}_i}(q \ldots, \dot{q} \ldots, t)$

$q, \dot{q}, t \rightarrow q, p, t$ requires Legendre transformation.

Consider $f(x,y)$ $\quad df = u\,dx + v\,dy$

$u = \left(\frac{\partial f}{\partial x}\right)_y \qquad v = \left(\frac{\partial f}{\partial y}\right)_x$

Let $g = f - ux = g(u, y)$

$$dg = df - u\,dx - x\,du = v\,dy - x\,du$$

u, y now primary variables

<u>Thermodynamic Diversion</u>

1st Law: $dU = dQ - dW$

gas undergoing reversible process

$$dU = T\,dS - p\,dV \qquad U = U(S,V)$$

$$T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

$S,V \rightarrow S,P \qquad H = U + PV \qquad dH = dU + P\,dV + V\,dP$

$$dH = T\,dS + V\,dP \qquad H = H(S,P)$$

$$T = (\partial H/\partial S)_P \qquad V = (\partial H/\partial P)_S$$

Also: $S,V \rightarrow T,V \qquad F = U - TS$

$$dF = dU - T\,dS - S\,dT = -S\,dT - P\,dV$$

$F = F(T,V)$ Helmholtz Free Energy

$S,P \rightarrow T,P \qquad G = H - TS \qquad dG = dH - T\,dS - S\,dT$

$$dG = -S\,dT + V\,dP \qquad G = G(P,T)$$

Gibbs Free Energy

Back to mechanics: $L = L(q_i, \dot{q}_i, t)$

$$dL = \frac{\partial L}{\partial q_i}\,dq_i + \frac{\partial L}{\partial \dot{q}_i}\,d\dot{q}_i + \frac{\partial L}{\partial t}\,dt$$

$$p_i \equiv \partial L/\partial \dot{q}_i$$

$$\frac{d}{dt}(p_i) - \frac{\partial L}{\partial q_i} = 0$$ Lagrangian equation

$$\therefore \dot{p}_i = \frac{\partial L}{\partial q_i}$$

$$\therefore dL = \dot{p}_i\, dq_i + p_i\, d\dot{q}_i + \frac{\partial L}{\partial t}\, dt$$

Now define $H(q,p,t) = \dot{q}_i p_i - L(q,\dot{q},t)$

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

$$= \dot{q}_i\, dp_i + p_i\, d\dot{q}_i - dL$$

$$= \dot{q}_i\, dp_i + p_i\, d\dot{q}_i - \dot{p}_i\, dq_i - p_i\, d\dot{q}_i - \partial L/\partial t\, dt$$

$$dH = \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - (\partial L/\partial t)\, dt$$

$$\boxed{\therefore \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad -\dot{p}_i = \frac{\partial H}{\partial q_i} \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}}$$

$2n+1$ "canonical equations"

Note: H & h (energy function) are the same numerically but have different independent variables

FORMAL PROCEDURE TO DETERMINE HAMILTONIAN

1. Choose generalized coordinates & write Lagrangian $L = L(q_i, \dot{q}_i, t) = T - V$
2. Define conjugate momenta $p_i = \partial L/\partial \dot{q}_i$
3. Form the Hamiltonian, most generally by $H = \dot{q}_i p_i - L$. At this stage, we have some

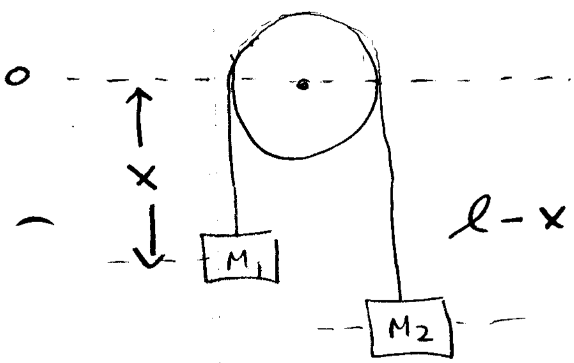
mixed function of q_i, $\dot{q}_i$, p_i, and t

4. Get rid of the $\dot{q}_i$ by inverting $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

5. $\mathcal{H}$ then solely a function of q, p, t + canonical equations can be utilized.

SIMPLE PROBLEMS

1. ATWOOD'S MACHINE



$$V = -M_1 g x - M_2 g(l-x)$$

$$T = \frac{1}{2}(M_1+M_2)\dot{x}^2$$

$$L = T - V$$

1 generalized coordinate

(1.) $L = \frac{1}{2}(M_1+M_2)\dot{x}^2 + M_1 g x + M_2 g(l-x)$

(2.) $p_x \equiv \dfrac{\partial L}{\partial \dot{x}} = \dot{x}(M_1+M_2)$

(3.) $\mathcal{H} = \dot{x}p_x - L = \dot{x}p_x - \frac{1}{2}(M_1+M_2)\dot{x}^2 - M_1 g x - M_2 g(l-x)$

(4.) $\dot{x} = p_x / M_1+M_2 \Rightarrow$

$$\mathcal{H} = \frac{p_x^2}{M_1+M_2} - \frac{1}{2}\frac{(M_1+M_2)p_x^2}{(M_1+M_2)^2} - M_1 g x - M_2 g(l-x)$$

$$\mathcal{H} = \underbrace{\frac{1}{2}\frac{p_x^2}{M_1+M_2}}_{T} \underbrace{- M_1 g x - M_2 g(l-x)}_{V}$$

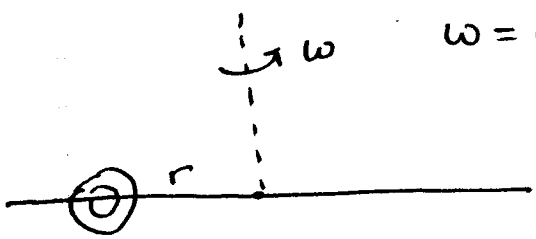
$$\frac{p_{x_1}^2}{2m_1} + \frac{p_{x_2}^2}{2m_2} \leftarrow T + V$$

5. $\dot{x} = \dfrac{\partial \mathcal{H}}{\partial P_x}$ $\qquad -\dot{P}_x = \dfrac{\partial \mathcal{H}}{\partial x}$

$\dot{x} = \dfrac{P_x}{M_1+M_2}$ $\qquad -\dot{P}_x = -M_1 g + M_2 g$

$$\ddot{x} = \frac{\dot{P}_x}{M_1+M_2} = \frac{(M_1 - M_2)\, g}{M_1+M_2}$$

2. <u>BEAD PROBLEM</u>



$\omega = \dot{\theta}$

1) $L = T = \frac{m}{2}\left(\dot{r}^2 + r^2\omega^2\right)$

2. $p_r = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$

r : generalized coordinate

$\boxed{\begin{array}{r}3.\\+4.\end{array}}$ $\mathcal{H} = \dot{r}p_r - L = \dfrac{p_r^2}{m} - \dfrac{m}{2}\dfrac{p_r^2}{m^2} - \dfrac{m}{2}r^2\omega^2$

$\mathcal{H} = \frac{1}{2}\dfrac{p_r^2}{m} - \dfrac{m}{2}r^2\omega^2$ $\qquad$ NOT SIMPLE T+V FORM (rheonomous)

5. $\dot{r} = \dfrac{\partial \mathcal{H}}{\partial p_r} = \dfrac{p_r}{m}$ $\qquad \dot{p}_r = -\dfrac{\partial \mathcal{H}}{\partial r} = m r \omega^2$

$$\boxed{\ddot{r} = r\omega^2}$$

3. <u>Free Particle In Polar Coordinates</u>

1. $L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$ $\qquad$ 2 generalized coordinates

2. $P_r = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\qquad P_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\quad (L_z)$
cyclic

3. $\mathcal{H} = \dot{r} p_r + \dot{\theta} p_\theta - L$

$$\mathcal{H} = \dot{r} p_r + \dot{\theta} p_\theta - \frac{m}{2}\dot{r}^2 - \frac{m}{2} r^2 \dot{\theta}^2$$

4. $\mathcal{H} = \dfrac{p_r^2}{m} + \dfrac{p_\theta^2}{mr^2} - \dfrac{m}{2}\dfrac{p_r^2}{m^2} - \dfrac{mr^2}{2}\dfrac{p_\theta^2}{m^2 r^4}$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} = T$$

5. $\dot{r} = \dfrac{\partial \mathcal{H}}{\partial p_r} = \dfrac{p_r}{m}$ $\qquad -\dot{p}_r = \dfrac{\partial \mathcal{H}}{\partial r} = -\dfrac{p_\theta^2}{mr^3}$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad -\dot{p}_\theta = \frac{\partial \mathcal{H}}{\partial \theta} = 0$$

(cyclic)

p_θ (L_z) constant of the motion

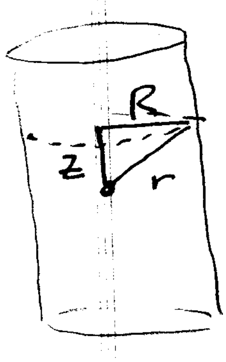
$$\dot{\theta} = p_\theta / mr^2 \qquad \ddot{r} = \frac{\dot{p}_r}{m} = p_\theta^2 / m^2 r^3$$

$$\frac{d^2 r}{dt^2} = \frac{p_\theta^2}{m^2 r^3}$$

4. PARTICLE CONSTRAINED TO MOVE ON THE SURFACE OF A CYLINDER

MARION P. 245

$$V = \tfrac{1}{2} k r^2 = \tfrac{1}{2} k (R^2 + z^2)$$



$\boxed{x^2 + y^2 = R^2}$ constraint

$$T = \frac{m}{2}\left(R^2 \dot{\theta}^2 + \dot{z}^2\right)$$

(1) $L = T - V = \dfrac{m}{2}\left(R^2 \dot{\theta}^2 + \dot{z}^2\right) - \dfrac{k}{2}\left(R^2 + z^2\right)$

(2) $P_\theta = \dfrac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \qquad P_z = \dfrac{\partial L}{\partial \dot z} \quad m\dot z$

cyclic coordinate (constant)

(3) $\mathcal{H} = \dot\theta P_\theta + \dot z P_z - L = \dot\theta P_\theta + \dot z P_z$

$$- \frac{m}{2}\left(R^2\dot\theta^2 + \dot z^2\right) + \frac{k}{2}\left(R^2 + z^2\right)$$

(4) $$\mathcal{H} = \underbrace{\frac{P_\theta^2}{2mR^2} + \frac{P_z^2}{2m}}_{T} + \underbrace{\frac{1}{2}kz^2 + \frac{1}{2}kR^2}_{V}$$

(5) $\dot\theta = \dfrac{\partial \mathcal{H}}{\partial P_\theta} = \dfrac{P_\theta}{mR^2} \qquad \dot P_\theta = -\dfrac{\partial \mathcal{H}}{\partial \theta} = 0$

constant

$\dot z = \dfrac{\partial \mathcal{H}}{\partial P_z} = \dfrac{P_z}{m} \qquad \dot P_z = -\dfrac{\partial \mathcal{H}}{\partial z} = -kz$

$\ddot z = \dot P_z / m = -\dfrac{k}{m} z$

A Brief Diversion

The canonical equations give us the time dependence of P_q & q. What about more general parameters that depend on P and q? $u = u(p, q, t)$

How does u evolve?

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial u}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial u}{\partial t}$$