Review: Canonical Transformation

$q, p, t \longrightarrow Q, P, t$ phase space (to simplify problem)

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \qquad \dot{Q} = \frac{\partial K}{\partial P} \qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

$$\boxed{p_i \dot{q}_i - \mathcal{H} = P_i \dot{Q}_i - K + \frac{dF}{dt}}$$

Some Simple Generating Functions

$$F = F_1(q, Q, t) \Rightarrow p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$K = \mathcal{H} + \frac{\partial F_1}{\partial t}$$

$$F = F_2(q, P, t) - Q_i P_i$$

$$\Rightarrow \quad p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = \mathcal{H} + \frac{\partial F_2}{\partial t}$$

$$F = F_3(p, Q, t) + q_i p_i$$

$$\Rightarrow \quad q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i} \qquad K = \mathcal{H} + \frac{\partial F_3}{\partial t}$$

$$F = F_4(p, P, t) + q_i p_i - Q_i P_i$$

$$q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i} \qquad K = \mathcal{H} + \frac{\partial F_4}{\partial t}$$

Direct Approach To Restricted Canonical Transformations

(no explicit time dependence).

$$Q_i = Q_i(q, p) \qquad P_i = P_i(q, p)$$

For a <u>restricted</u> transformation $K = \mathcal{H}$ since $\frac{\partial F}{\partial t} = 0$

Consider

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j} \underbrace{\dot{q}_j}_{\frac{\partial \mathcal{H}}{\partial p_j}} + \frac{\partial Q_i}{\partial p_j} \underbrace{\dot{p}_j}_{-\frac{\partial \mathcal{H}}{\partial q_j}}$$

$$\therefore \dot{Q}_i = \frac{\partial Q_i}{\partial q_j} \frac{\partial \mathcal{H}}{\partial p_j} + - \frac{\partial Q_i}{\partial p_j} \frac{\partial \mathcal{H}}{\partial q_j}$$

but

$$\dot{Q}_i = \frac{\partial \mathcal{H}}{\partial P_i} = \frac{\partial \mathcal{H}}{\partial p_j} \frac{\partial p_j}{\partial P_i} + \frac{\partial \mathcal{H}}{\partial q_j} \frac{\partial q_j}{\partial P_i}$$

$$\therefore \boxed{\frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{\partial P_i} \quad + \quad \frac{\partial Q_i}{\partial p_j} = - \frac{\partial q_j}{\partial P_i}}$$

square matrices

Starting from $\dot{P}_i$, we get that   Symplectic version

$$\boxed{\frac{\partial P_i}{\partial q_j} = - \frac{\partial p_j}{\partial Q_i} \qquad \frac{\partial P_i}{\partial p_j} = \frac{\partial q_j}{\partial Q_i}}$$

Symplectic Version

2n column matrix $\eta = \begin{bmatrix} q_i\text{'s} \\ \vdots \\ p_i\text{'s} \end{bmatrix}$ $\qquad \zeta = \begin{bmatrix} Q_i\text{'s} \\ \vdots \\ P_i\text{'s} \end{bmatrix}$

Jacobian matrix $\quad M_{ij} = \dfrac{\partial \zeta_i}{\partial \eta_j}$

$$M J = J \tilde{M}^{-1}$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Example : Direct

$$Q = \log\left[\frac{1}{q}\sin p\right] \qquad P = q\cot p$$

must invert $q = q(Q,P)$ $\quad p = p(Q,P)$

$$e^{Q} = \frac{1}{q}\sin p \qquad P = q\cos p/\sin p$$

$$e^{Q}P = \cos p \Rightarrow p = \cos^{-1}\left[e^{Q}P\right]$$

$$q = \frac{\sin p}{e^{Q}} = e^{-Q}\sqrt{1-(e^{Q}P)^2}$$

$$p = \cos^{-1}\left[e^{Q}P\right]$$

$$\frac{\partial Q}{\partial q} = \frac{-1/q^2\sin p}{1/q\sin p} = -1/q$$

$$\frac{\partial p}{\partial P} = -\frac{1}{\sqrt{1-(e^{Q}P)^2}}\,e^{Q} = -\frac{e^{Q}}{q\,e^{Q}} = -\frac{1}{q} \checkmark$$

$$\frac{\partial Q}{\partial p} = \frac{1/q\cos p}{1/q\sin p} = \cot p$$

$$-\frac{\partial q}{\partial P} = e^{-Q}\,\frac{1}{2}\left(1-e^{2Q}P^2\right)^{-1/2}e^{2Q}\,2P$$

$$= \frac{e^{Q}P}{\sqrt{1-(e^{Q}P)^2}} = \frac{\cos p}{\sin p} = \cot p \checkmark$$

etc.

Indirect Approach

find generating function of the old & the new coordinate

$$e^{Q} = \frac{1}{q} \sin p \qquad \boxed{q = e^{-Q} \sin p}$$

$$P = q \cot p = e^{-Q} \sin p \frac{\cos p}{\sin p} = \boxed{e^{-Q} \cos p = P}$$

p, Q independent variables

Suggests:

$$F = F_3(p, Q) + qP$$

$$q = -\frac{\partial F_3}{\partial p} = e^{-Q} \sin p$$

$$P = -\frac{\partial F_3}{\partial Q} = e^{-Q} \cancel{\sin p} \cos p$$

$$F_3 = + e^{-Q} \cos p$$

The 1-D Harmonic Oscillator

$$\mathcal{H} = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2 \qquad \omega^2 = k/m$$

If we use $F_1 = qQ \quad Q = p \quad P = -q$

$$K = \mathcal{H} = \frac{1}{2m} Q^2 + \frac{1}{2} m \omega^2 P^2 \quad \text{not very helpful}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = m\omega^2 P \qquad \dot{P} = -\frac{\partial K}{\partial Q} = -Q/m$$

$$\ddot{Q} = m\omega^2 \dot{P} = m\omega^2 \left(-Q/_m \right) = -\omega^2 Q$$

$$\ddot{P} = -\frac{1}{m}\dot{Q} = -\frac{1}{m} m\omega^2 P = -\omega^2 P$$

Normally: $\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = P/m \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q$

$$\ddot{q} = \dot{p}/m = -\omega^2 q \quad \text{etc.}$$

Now, let's consider a more useful canonical transformation, one designed to find cyclic coordinates:

$$\mathcal{H} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2 \right)$$

Suppose we could find a canonical transformation such that

$$p = f(P)\cos Q \qquad q = \frac{f(P)}{m\omega}\sin Q$$

then $K = H = \frac{f^2(P)}{2m}\left[\cos^2 Q + \sin^2 Q \right] = \frac{f^2(P)}{2m}$

with Q cyclic.

Try $F = F_1 = \frac{m\omega}{2} q^2 \cot Q$ can work backwards for Corason

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q$$

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{m\omega}{2} q^2 \frac{\partial \cot Q}{\partial Q} = \frac{m\omega}{2} q^2 \frac{1}{\sin^2 Q}$$

$$q^2 = \frac{2P}{m\omega}\sin^2 Q \implies q = \sqrt{\frac{2P}{m\omega}}\sin Q$$

then $p = m\omega q \cot Q = m\omega \sqrt{\frac{2P}{m\omega}} \sin Q \frac{\cos Q}{\sin Q}$

$$p = \sqrt{2Pm\omega}\cos Q$$

but we need: $p = f(P)\cos Q \qquad q = \frac{f(P)}{m\omega}\sin Q$

From p equations: $f(P) = \sqrt{2Pm\omega}$

From q equations: $\frac{f(P)}{m\omega} = \sqrt{\frac{2P}{m\omega}}$ ✓

$\therefore \quad K = H = \frac{2Pm\omega}{2m} = P\omega$

H cyclic in $Q \Rightarrow P$ a constant of the motion.
In fact
$$P = H/\omega = E/\omega$$

Now: $\dot{Q} = \frac{\partial K}{\partial P} = \omega \Rightarrow Q = \omega t + \alpha$

Solution: $P = E/\omega \qquad Q = \omega t + \alpha$

In terms of normal q and p:

$$q = \frac{f(P)}{m\omega}\sin Q = \sqrt{\frac{2P}{m\omega}}\sin Q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha)$$

$$p = f(P)\cos Q = \sqrt{2Pm\omega}\cos Q = \sqrt{2mE}\cos(\omega t + \alpha)$$

We have so far used transformation to find cyclic coordinates. There is an alternative method: to seek a canonical transformation from q, p at time t to a set of constant coordinates, specifically q_0, p_0 at time $t=0$. (time-dependent) $\quad q, p, t \rightarrow q_0, p_0, t=0$

The equations of transformation:

$$q = q(q_0, p_0, t) \qquad p = p(q_0, p_0, t)$$

are exactly the solution of the problem!!!

10. Hamilton - Jacobi Theory

To obtain new variables constant in time, set $K=0$.

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0 \qquad -\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0$$

but $K = \mathcal{H} + \dfrac{\partial F}{\partial t}$ too general ?

$$K=0 \implies \mathcal{H}(q,p,t) + \partial F/\partial t = 0$$

It is convenient to use a generating function of the second type:

$$F = F_2(\underbrace{q}_{\text{old coordinate}}, \underbrace{P}_{\text{new constant momentum}}, t) - \underbrace{Q_i P_i}_{\text{no time dependence}}$$

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$\boxed{n+1 \text{ variables } q_1 \cdots q_n ; t}$$

$$\boxed{\therefore \mathcal{H}\left(q_1 \cdots q_n ; \frac{\partial F_2}{\partial q_1} \cdots \frac{\partial F_2}{\partial q_n} ; t\right) + \frac{\partial F_2}{\partial t} = 0}$$

Hamilton - Jacobi Equation

can be thought of as a partial differential equation for the desired F_2, with $n+1$ variables $q_1 \cdots q_n ; t$

. solution for F_2 is denoted S and known as Hamilton's Principal Function.

Ignore, for a moment, how HJ equation was obtained!

Suppose there exists a solution to HJ of the form

$$S = S(q_1, \cdots q_n ; \underbrace{\alpha_1 \cdots \alpha_{n+1}}_{n+1 \text{ constants of integration}}, t)$$

"complete solution"

obtained in integrating $n+1$ partial derivatives

But S does NOT appear in HJ: only partials do.

If S is a solution, so is $S + \alpha$, where α is a constant. But a constant term is irrelevant for a generating function since only derivatives are important for canonical transformations.

$$\therefore \quad S = S(q_1 \ldots q_n \, ; \underbrace{\alpha_1 \ldots \alpha_n}_{\text{none solely additive}}, t) \text{ suffices}$$

Thus, the "complete solution" tallies with the generating function.

$$F = F_2(q, P, t) - Q_i P_i$$

and it makes sense to equate $P_i = \alpha_i$, new constant momenta.

For F_2 functions: $\quad p_i = \dfrac{\partial F_2}{\partial q_i} = \dfrac{\partial S}{\partial q_i}(q, \alpha, t) \qquad (1)$

$$Q_i \equiv B_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial S}{\partial \alpha_i}(q, \alpha, t) \quad (2)$$

<u>Initial Conditions</u> (once S is known)

Start with (1): set $t = t_0$
invert to yield α_i in terms of p_0, q_0's

Then go to (2) evaluate rhs at $t = t_0$
$\therefore$ B_i given in terms of α_i, q_0's.

Then for any t invert (2) to give
$q_j = q_j(\alpha, B, t)$

(3) back to equation (1). $\qquad (\alpha, B) \rightarrow (p_0, q_0)$
$p_i = p_i(\alpha, B, t)$

Now look at total time derivative of S.

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \qquad S = S(q, \underbrace{\alpha}_{\text{constant}}, t)$$

but, from the HJ equation: $\mathcal{H} + \frac{\partial S}{\partial t} = 0$, we have

$\partial S/\partial t = -\mathcal{H}$ Also $p_i = \partial S/\partial q_i$ so that

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i - \mathcal{H} = p_i \dot{q}_i - \mathcal{H}$$

Since $\mathcal{H} = p_i \dot{q}_i - L \Rightarrow L = p_i \dot{q}_i - \mathcal{H}$

so $\frac{dS}{dt} = L \qquad S = \int L\, dt$

equations of motion must be solved to be useful

Method II The W Approach (10.3 in detail)

Now, suppose $\mathcal{H}$ does not depend explicitly on time:

HJ equation $\partial S/\partial t = -\mathcal{H} \Rightarrow \frac{\partial S}{\partial t}$ also does not explicitly on time

Hence we can write $S(q, \alpha, t) = W(q, \alpha) - \alpha t$

$\frac{\partial S}{\partial t} = -\alpha$ where $\alpha = \mathcal{H} \neq P$'s

separate constant

W : Hamilton's characteristic function

$$\frac{dW}{dt} = \frac{\partial W}{\partial q_i}\dot{q}_i \qquad p_i = \frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i}$$

$$\frac{dW}{dt} = p_i \dot{q}_i \qquad W = \int p_i \dot{q}_i\, dt = \int p_i\, dq_i$$

"action"

Solution In Terms of W can bypass α_i, β_i in terms of α from $S = W - \alpha t$

Now let's look at the HJ Equation:

$$\mathcal{H}\left(q, \frac{\partial W}{\partial q}\right) - \alpha = 0$$

$$\boxed{\mathcal{H}\left(q, \frac{\partial W}{\partial q}\right) = \alpha \quad \underset{\text{"normally"}}{(E)}}$$

restricted HJ eq.

The Harmonic Oscillator (1-D)

$$\mathcal{H} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \qquad p = \frac{\partial W}{\partial q} = \frac{\partial S}{\partial q}$$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

$$\left(\frac{\partial W}{\partial q}\right)^2 = 2m\alpha - m^2\omega^2 q^2$$

$$\frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2} \qquad \alpha \text{ constant}$$

$$W = \int \underbrace{\left\{2m\alpha - m^2\omega^2 q^2\right\}^{1/2}}_{P} dq$$

$$S = \int \left\{2m\alpha - m^2\omega^2 q^2\right\}^{1/2} dq - \alpha t$$

$$\beta' = \frac{\partial S}{\partial \alpha} = \int \frac{1}{2}\left\{2m\alpha - m^2\omega^2 q^2\right\}^{-1/2} 2m\, dq - t$$

$$\left(Q = \frac{\partial S}{\partial P}\right) \quad \beta' + t = \frac{m}{\sqrt{2m\alpha}} \int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}}$$

Let $x = \sqrt{\frac{m\omega^2}{2\alpha}}\, q$ $\quad \beta' + t = \sqrt{\frac{m}{2\alpha}}\sqrt{\frac{2\alpha}{m\omega^2}} \int \frac{dx}{\sqrt{1-x^2}}$

$$\beta' + t = \frac{1}{\omega} \sin^{-1} x \qquad x = \sin[\omega(t + \beta')]$$

$$\beta \equiv \omega \beta' \qquad \sqrt{\frac{m\omega^2}{2\alpha}}\, q = \sin[\omega t + \beta]$$

$$q = \sqrt{2\alpha/m\omega^2}\, \sin[\omega t + \beta]$$

$$p = \sqrt{2m\alpha - m^2\omega^2 q^2}$$

$$p = \sqrt{2m\alpha - m^2\omega^2 \frac{2\alpha}{m\omega^2} \sin^2(\omega t + \beta)}$$

$$= \sqrt{2m\alpha\,(1 - \sin^2[\omega t + \beta])}$$

$$p = \sqrt{2m\alpha}\, \cos(\omega t + \beta)$$

So: $q, p \longleftrightarrow \alpha, \beta$ constant variables NOT p_0, q_0 really

energy phase angle

<u>Initial Conditions</u>

$$q^2 = \frac{2\alpha}{m\omega^2} \sin^2(\omega t + \beta)$$

$$p^2 = 2m\alpha \cos^2(\omega t + \beta)$$

at any time $m^2\omega^2 q^2 + p^2 = 2m\alpha$ including $t = t_0$

so $p_0^2 + m\omega^2 q_0^2 = 2m\alpha$, but:

Initial Conditions, etc.

$t = 0$ $\qquad q_0^2 = \dfrac{2\alpha}{m\omega^2} \sin^2 \beta$

$$p_0^2 = 2m\alpha \cos^2 \beta$$

$$q_0^2/p_0^2 = \frac{2\alpha}{m\omega^2} \frac{}{2m\alpha} \tan^2 \beta$$

$$m\omega\, q_0/p_0 = \tan\beta$$

$$q_0 = 0 \iff \beta = 0$$

S generator of $K = 0$ to $\alpha = E$, $\beta =$ phase

Also:

$$S = \sqrt{2m\alpha} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} - \alpha t$$

$$= 2\alpha \int \underbrace{\left[\cos^2 [\omega t + \beta] - \frac{1}{2}\right]}_{L} dt$$

2-D Anisotropic Harmonic Oscillator

$$\mathcal{H} = E = \frac{1}{2m} \left(p_x^2 + p_y^2 + m^2 \omega_x^2 x^2 + m^2 \omega_y^2 y^2 \right)$$

(type 2 gen f's) $\quad S = S(x, y, \alpha, \alpha_y, t) = \underbrace{W_x (x, \alpha) + W_y (y, \alpha_y)}_{W(x,y,\alpha,\alpha_y)} - \alpha t$

$$\frac{1}{2m}\left[\underbrace{\left(\frac{\partial W}{\partial x}\right)^2}_{(p_x^2)} + m^2\omega_x^2 x^2 + \left(\frac{\partial W}{\partial y}\right)_z^2 + m^2 \omega_y^2 y^2 \right] = \alpha$$

(HJ eq)

x & y parts separate.

$$\therefore \frac{1}{2m}\left[\left(\frac{\partial W}{\partial y}\right)^2 + m^2 \omega_y^2 y^2\right] = \alpha_y$$

$$\alpha_x = \alpha - \alpha_y \Rightarrow \quad \frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + m^2 \omega_x^2 x^2\right] = \alpha - \alpha_y = \alpha_x$$

As in the 1-D case:

$$x = \sqrt{\frac{2\alpha_x}{m\omega_x^2}} \sin(\omega_x t + \beta_x)$$

$$y = \sqrt{\frac{2\alpha_y}{m\omega_y^2}} \sin(\omega_y t + \beta_y)$$

$$p_x = \sqrt{2m\alpha_x} \cos(\omega_x t + \beta_x)$$

$$p_y = \sqrt{2m\alpha_y}\ \overset{\cos}{\cancel{\sin}}\,(\omega_y t + \beta_y)$$

$$E = \alpha_x + \alpha_y = \alpha$$

2-D Isotropic Oscillator in Polar Coordinates

$k_x = k_y \qquad \omega_x = \omega_y = \omega$

$x = r\cos\theta \qquad y = r\sin\theta \qquad L = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) - V$

$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1} y/x$

$p_r = m\dot r \qquad p_\theta = mr^2\dot\theta$

$$\mathcal{H} = E = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + m^2\omega^2 r^2\right)$$

cyclic in θ $\therefore$ p_θ constant

$$S(r,\theta,\alpha,\alpha_\theta,t) = W_r(r,\alpha) + W_\theta(\theta,\alpha_\theta) - \alpha t$$

$$p_\theta = \frac{\partial S}{\partial\theta} = \frac{\partial W_\theta}{\partial\theta} = \alpha_\theta \text{ constant}$$

$$\Rightarrow W_\theta = \theta\alpha_\theta$$

$$E = \mathcal{H} = \frac{1}{2m}\left[p_r^2 + \frac{\alpha_\theta^2}{r^2} + m^2\omega^2 r^2 \right]$$

W_r :

$$HJ \Rightarrow \frac{1}{2m}\left[\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} + m^2\omega^2 r^2 \right] = \alpha$$

Easiest Solution (unclean)

Take solutions for x, y, p_x, p_y & transfm: $(\beta_x \overset{?}{=} \beta_y)$

$$r = \sqrt{\frac{2\alpha}{m\omega^2}}\sqrt{\sin^2\omega t + \sin^2(\omega t + \beta)} \qquad p_r = m\dot{r}$$

?

$$\theta = \tan^{-1}\left[\frac{\sin\omega t}{\sin(\omega t + \beta)}\right] \qquad p_\theta = m r^2 \dot{\theta}$$

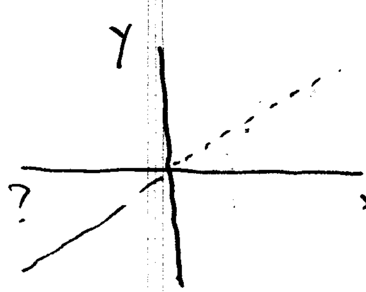
[not general enough]

$r, \theta, p_r, p_\theta \Rightarrow \alpha, \alpha_\theta, \beta, ?$

Two Limiting Cases

(i) $\beta = 0$ $\quad r = \sqrt{\frac{4\alpha}{m\omega^2}}\sin\omega t \qquad p_r = \sqrt{4m\alpha}\cos\omega t$
$(m\dot{r})$

$$\theta = \tan^{-1} 1 = \pi/4 \qquad p_\theta = 0$$



~~sin t~~

(ii) $\beta = \pi/2$

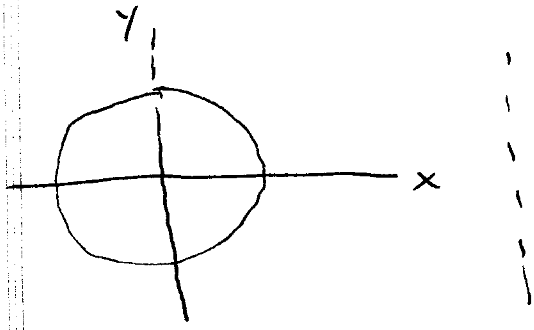
$$r = \sqrt{\frac{2\alpha}{m\omega^2}} \cancel{\frac{\sin\omega t}{\sin(\omega t + \pi/2)}}$$

$$\times \sqrt{\sin^2\omega t + \sin^2(\omega t + \pi/2)}$$

$$\sin(\omega t + \pi/2) = \cos\omega t$$

$$\therefore\ r = r_0 = \sqrt{\frac{2\alpha}{m\omega^2}} \qquad \theta = \tan^{-1}\tan\omega t = \omega t$$

$$p_r = 0 \qquad p_\theta = m r^2 \omega$$



circular motion

$0 < \beta < \pi/2$

ellipse