9. Canonical Transformations

Consider a situation where the Hamiltonian is a constant of the motion and where all coordinates q_i are cyclic.

$\therefore \quad p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ all constant $\quad p_i = \alpha_i$

$$\mathcal{H} = \mathcal{H}(p_i) = \mathcal{H}(\alpha_1, \alpha_2 \ldots \alpha_n)$$

$\dot{q}_i = \dfrac{\partial \mathcal{H}(p)}{\partial p_i} = \dfrac{\partial \mathcal{H}}{\partial \alpha_i} = \omega_i$, independent of t and q_i

$\omega_i = \omega_i(\alpha' s)$ + hence constant

Solution: $\quad q_i = \omega_i t + \beta_i \quad$ Is this only of academic interest?

$\mathcal{H} = p^2/2m \quad \dot{q} = \dfrac{\partial \mathcal{H}}{\partial p} = p/m \equiv \omega$

Remember that a given system can generally be handled by different sets of generalized coordinates.

e.g. consider a particle in a plane

$q_1 = x \quad q_2 = y$ cartesian or $q_1 = r \quad q_2 = \theta$ polar

Depending upon V, one set may be more convenient. For central forces, for example, the coordinate θ is cyclic.

Thus, there may well be a particular choice for which all coordinates are cyclic, but such a set is far from obvious. We need a procedure for transforming from one set of generalized coordinates to another.

Transformation must include conjugate momenta.

$$Q_i = Q_i(q, p, t) \qquad P_i = P_i(q, p, t)$$

"point transformation of phase space"

Q and P must remain canonical, of course. This requirement will be met if there exists a K:

$K(Q,P,t)$: transformed Hamiltonian "Kamiltonian"

such that $\dot{Q}_i = \dfrac{\partial K}{\partial P_i}$ $\quad \dot{P}_i = -\dfrac{\partial K}{\partial Q_i}$

This transformation is to be problem-independent, but general for a given number of degrees of freedom.

Need to satisfy modified Hamilton's Principle:

$$\delta \int_{t_1}^{t_2} \left[P_i \dot{Q}_i - K(Q,P,t) \right] dt = 0$$

but also $\delta \displaystyle\int_{t_1}^{t_2} \left[p_i \dot{q}_i - \mathcal{H}(q,p,t) \right] dt = 0$

(fixed end-points). Both statements satisfied if

$$\underset{\text{constant}}{\lambda} \cdot (p_i \dot{q}_i - \mathcal{H}) = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

no proof given

$$\int_{t_1}^{t_2} \frac{dF}{dt}\, dt = F(t_2) - F(t_1)$$ no variation at end-points

$F = F(q,p,t)$ or $F(Q,P,t)$ must have continuous 2nd derivatives and is known as a "generating function" λ: constant with respect to coordinates and time.

Scale Transformation

$Q_i' = \mu q_i \quad P_i' = \nu p_i$

$'$: specific transformation

then $K'(Q', P') = \mu\nu\, \mathcal{H}(q, p)$

because $$\frac{\partial K'}{\partial P_i'} = \frac{\partial K'}{\partial p_i}\frac{\partial p_i}{\partial P_i'} = \frac{\partial K'}{\partial p_i}\frac{1}{\nu} = \mu\frac{\partial \mathcal{H}}{\partial p_i}$$
$$= \mu \dot{q}_i = \dot{Q}_i'$$

and $$\frac{\partial K'}{\partial Q_i'} = \frac{\partial K'}{\partial q_i}\frac{1}{\mu} = \nu\frac{\partial \mathcal{H}}{\partial q_i} = \nu(-\dot{p}_i)$$
$$= -\dot{P}_i'$$

also $$P_i'\dot{Q}_i' - K' = \mu\nu(p_i\dot{q}_i - \mathcal{H})$$

which means that $\lambda = \mu\nu$. The scale change can be removed in a transformation:

$$P_i' = \nu P_i \qquad Q_i' = \mu Q_i \qquad K' = \mu\nu K$$

$$\Rightarrow \quad p_i\dot{q}_i - \mathcal{H} = P_i\dot{Q}_i - K + \underbrace{\frac{dF}{dt}}_{\text{otherwise no transformation}}$$
$(\lambda = 1)$

$\lambda \neq 1$ extended canonical transformation
$\lambda = 1$ canonical transformation
no time dependence & $\lambda = 1$ restricted canonical transformation

$$\lambda = 1 \quad p_i\dot{q}_i - \mathcal{H} = P_i\dot{Q}_i - K + \underbrace{\frac{dF}{dt}}$$

F useful as generating function if it contains independent variables from old & new sets.

Some Simple Generating Functions

(1) $F = F_1(q, Q, t)$

$$p_i \dot{q}_i - \mathcal{H} = P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i$$

(0) $$\dot{q}_i\left(p_i - \frac{\partial F_1}{\partial q_i}\right) - \dot{Q}_i\left(P_i + \frac{\partial F_1}{\partial Q_i}\right) - \mathcal{H} + K - \frac{\partial F_1}{\partial t} = 0$$

2 sets of linearly independent coordinates

$$\therefore \underbrace{p_i = \frac{\partial F_1}{\partial q_i} \quad (1)}_{n \text{ eq's}} \quad \underbrace{P_i = -\frac{\partial F_1}{\partial Q_i} \quad (2)}_{n \text{ eq's.}} \quad \underbrace{K = \mathcal{H} + \frac{\partial F_1}{\partial t}}_{1 \text{ eq.}} \quad (3)$$

$K = \mathcal{H}$ if no time dep.

Equations (1) yield p_i in terms of q_j, Q_j, t. Inversion yields Q_i in terms of q_j, p_j, t.

Equations (2) yield P_i in terms of q_j, Q_j, t or, with help of (1), in terms of q_j, p_j, t.

Then, q & p in $\mathcal{H}$ and q in $\partial F_1/\partial t$ are expressed in terms of Q, P and the two functions added to get K.

Other generating functions:

(2) $F = F_2(q, P, t) - Q_i P_i$

Suppose it is more convenient to write p in terms of q, P, t. Then want F_2 in terms of q and P. But must replace $\dot{Q}_i$ with $\dot{P}_i$ in (0)

$$p_i\dot{q}_i - \mathcal{H} = \cancel{P_i\dot{Q}_i} - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i$$

Show failure mode first w/o Q_iP_i

$$- \cancel{\dot{Q}_i P_i} - Q_i\dot{P}_i$$

$$= -Q_i\dot{P}_i - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i$$

$$\dot{q}_i\left[p_i - \frac{\partial F_2}{\partial q_i}\right] + \dot{P}_i\left[Q_i - \frac{\partial F_2}{\partial P_i}\right] - \mathcal{H} + K - \frac{\partial F_2}{\partial t} = 0$$

$$\therefore\ p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = \mathcal{H} + \frac{\partial F_2}{\partial t}$$

Two other analogous possibilities:

(3) $F = F_3(p, Q, t) + q_i p_i$

$$\Rightarrow q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i} \qquad K = \mathcal{H} + \frac{\partial F_3}{\partial t}$$

(4) $F = F_4(p, P, t) + q_i p_i - Q_i P_i$

$$\Rightarrow q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i} \qquad K = \mathcal{H} + \frac{\partial F_4}{\partial t}$$

TRIVIAL EXAMPLES

(1) $F_1 = q_i Q_i \qquad p_i = \frac{\partial F_1}{\partial q_i} = Q_i \qquad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i$

$$Q_i = p_i \qquad P_i = -q_i \qquad K = \underbrace{\mathcal{H}}_{\text{must substitute } q_i = -P_i,\ p_i = Q_i}$$

e.g. $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kq^2$

$$\rightarrow K = Q^2/2m + \frac{1}{2}kP^2$$