Physics 821 — Answers To Midterm — Spring 2007, Dr. Herbst

1. a) Contact moves at zero speed.

$$\therefore \underbrace{(b-a)\dot\theta}_{\text{c. of mass motion}} = \underbrace{a\dot\varphi}_{\text{roll}} \qquad \dot\varphi = (b-a)\dot\theta/a$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0 \qquad L = T_{cm} + T_{roll} - V \; ; \; V = -mg(b-a)\cos\theta$$

$$L = \frac{1}{2}m(b-a)^2\dot\theta^2 + \frac{1}{2}I\dot\varphi^2 + mg(b-a)\cos\theta$$

$$(b-a)\dot\theta = a\dot\varphi$$

$$(b-a)\theta = a\varphi \qquad \text{integrable to holonomic constraint}$$

$$L = \frac{m}{2}(b-a)^2\dot\theta^2 + \frac{1}{2}ma^2\left[\frac{b-a}{a}\right]^2\dot\theta^2 + mg(b-a)\cos\theta$$

$$L = m(b-a)^2\dot\theta^2 + mg(b-a)\cos\theta$$

$$\frac{\partial L}{\partial\dot\theta} = 2m(b-a)^2\dot\theta \qquad \frac{\partial L}{\partial\theta} = -mg(b-a)\sin\theta$$

$$\therefore \quad 2m(b-a)^2\ddot\theta + mg(b-a)\sin\theta = 0$$

$$\text{or } \ddot\theta = -\frac{g}{2(b-a)}\sin\theta$$

b)

$$\ddot\theta\dot\theta = -\frac{g}{2(b-a)}\sin\theta\,\dot\theta$$

$$\frac{1}{2}\frac{d}{dt}(\dot\theta^2) = +\frac{g}{2(b-a)}\frac{d}{dt}(\cos\theta)$$

$$\dot\theta^2 = \frac{g}{b-a}\cos\theta + K$$

$$t=0 \quad \dot\theta = 0 \Rightarrow K = -g/_{b-a}\cos\theta_0$$

$$\dot\theta^2 = g/_{b-a}\left[\cos\theta - \cos\theta_0\right]$$

$$\dot\theta = \pm\sqrt{\left(\frac{g}{b-a}\right)(\cos\theta - \cos\theta_0)}$$

1 c) $\frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0$ $\underset{\text{conserved}}{h} = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L$

$$h = 2m(b-a)^2 \dot{\theta}^2 - m(b-a)^2 \dot{\theta}^2 - mg(b-a)\cos\theta$$
$$h = T + V = E \text{ conserved}$$

$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2m(b-a)^2 \dot{\theta}^2$ is <u>NOT</u> conserved.

2. a) $E = \frac{1}{2} m\dot{r}^2 - \frac{k}{r} + \ell^2/2mr^2 = -\frac{k}{r} + \ell^2/2mr^2$

$\dot{r} = 0$ extrema $\quad r^2 E + kr - \ell^2/2m = 0 \quad (1)$

$r^2 + \frac{k}{E} r - \frac{\ell^2}{2mE} = 0 \qquad r_\pm = -\frac{k}{2E} \pm \frac{1}{2}\sqrt{\left(\frac{k}{E}\right)^2 + \frac{4\ell^2}{2mE}}$

$r_\pm = -\frac{k}{2E} \pm \frac{1}{2}\sqrt{(k/E)^2 + 2\ell^2/mE}$

i) $E > 0$ only r_+ since $r_- < 0$. (hyperbola)

ii) $E = 0$ (1) $\Rightarrow$ $r = \ell^2/2mk$ one solution (parabola)

iii) $E < 0$ if $\left[(k/E)^2 + 2\ell^2/mE\right] > 0$ two solutions since r_+ & r_- (+).

$(k/E)^2 > -2\ell^2/mE$

$\Rightarrow \frac{2\ell^2}{mE} > -(k/E)^2 \Rightarrow E > -\frac{mk^2}{2\ell^2}$

$\therefore \boxed{-\frac{mk^2}{2\ell^2} < E < 0}$ ellipse

$r_+ + r_- = 2a = -k/E \Rightarrow E = -k/2a$

b) $\frac{1}{2} m\dot{r}^2 = E - V(r) - \ell^2/2mr^2$

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \ell^2/2mr^2\right)} = \frac{dr}{dt}$$

$$\int_0^t dt' = \int_{r_{min}}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - V(r') - \ell^2/2mr'^2\right)}} = t(r)$$

3. a)

$$\left| \begin{matrix} \begin{bmatrix} a-\lambda & c \\ c & a-\lambda \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & b-\lambda \end{matrix} \right| = 0$$

$\lambda = b$ $\quad (a-\lambda)^2 = c^2 \quad a - \lambda = \pm c$

$\lambda = a \pm c$ $\quad$ 3 different roots asymmetric top

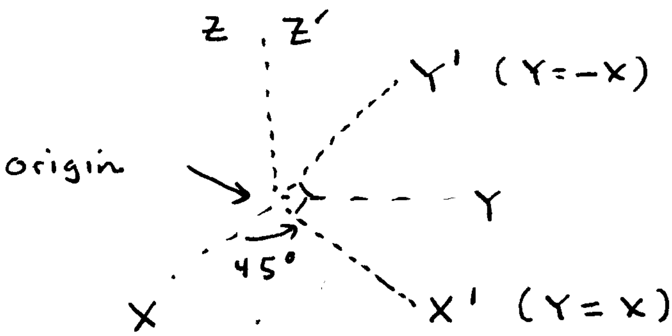
$\lambda = b \Rightarrow$ principal axis along Z

otherwise: $\quad (a-\lambda)X + cY = 0$

$cX + (a-\lambda)Y = 0$ $\quad$ same information

$\lambda_+ = a + c \qquad -cX + cY = 0 \Rightarrow Y = X$

$\lambda_- = a - c \qquad cX + cY = 0 \Rightarrow Y = -X$



b) $\varphi = 45°$

$$\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$