Physics 821 Spring 2007

Answers To HW 7

Goldstein et al. Ch. 6

(E 5)

$$\eta_i = C_k a_{ik} e^{-i\omega_k t} = a_{ik} \xi_k$$

$$\xi_k = C_k e^{-i\omega_k t}$$

$$\mathrm{Re}\, C_\ell = a_{j\ell} T_{jk} \eta_k(0) \qquad \mathrm{Im}\, C_\ell = \frac{1}{\omega_\ell} a_{j\ell} T_{jk} \dot{\eta}_k(0)$$

$\omega_1 = 0$ translation $\quad a_1 = \frac{1}{\sqrt{2m+M}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$

$\omega_2 = \sqrt{k/m}$ symm. stretch $\quad a_2 = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$

$\omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$ asymm. stretch $\quad a_3 = \begin{bmatrix} 1/\sqrt{2m(1+2m/M)} \\ -2/\sqrt{2M(2+M/m)} \\ 1/\sqrt{2m(1+2m/M)} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$

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(a) $\eta_1(0) = \eta_3(0) = 0 \quad \eta_2(0) = a_0$

$\dot{\eta}_i(0) = 0 \quad \therefore C$ <u>real</u>

Ignore translation since question asks for answer about center of mass!!

$$\therefore C_2 = a_{j2} T_{jk} \eta_k(0) = a_{j2} T_{j2} a_0 = \underbrace{a_{22}}_{0} \underbrace{T_{22}}_{M} a_0 = 0$$

$$T = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$C_3 = a_{j3} T_{jk} \eta_k(0) = a_{23} M a_0$$ only asymm stretch excited

$$\therefore \eta_1 = C_3 a_{13} e^{-i\omega_3 t} = \underbrace{a_{23} M a_0 a_{13}}_{\text{amplitude}} \exp[-i\omega_3 t]$$

$$\eta_2 = C_3 a_{23} e^{-i\omega_3 t} = a_{23} M a_0 a_{23} \exp[-i\omega_3 t]$$

$$\eta_3 = C_3 a_{33} e^{-i\omega_3 t} = a_{23} M a_0 a_{33} \exp[-i\omega_3 t]$$

Actual Motion: $\eta_i = \eta_i^{amp} \cos \omega_3 t$ (real)

Also: $\xi_2 = C_2 e^{-i\omega_2 t} = 0 \qquad \xi_3 = C_3 e^{-i\omega_3 t}$

$$= \underbrace{a_{23} M a_0}_{\text{amplitude}} e^{-i\omega_3 t}$$

(b) $\eta_i(0) = 0 \qquad \dot{\eta}_1(0) = \dot{\eta}_3(0) = 0 \qquad \dot{\eta}_2(0) = v_0$

$\mathrm{Re}\, C_\ell = 0$ Ignore translational mode $\mathrm{Im}\, C_2 = 0$

$$\mathrm{Im}\, C_3 = \frac{1}{\omega_3} a_{j3} T_{jk} \dot{\eta}_k(0) = \frac{1}{\omega_3} a_{23} M v_0$$

$$\therefore \eta_1 = i(\mathrm{Im}\, C_3) a_{13} e^{-i\omega_1 t} \qquad i = e^{i\pi/2}$$

$$\eta_1 = \underbrace{\frac{1}{\omega_3} a_{23} M v_0 a_{13}}_{\text{amplitude}} e^{-i(\omega_3 t - \pi/2)}$$

$$\eta_2 = \frac{1}{\omega_3} a_{23} M v_0 a_{23} e^{-i(\omega_3 t - \pi/2)}$$

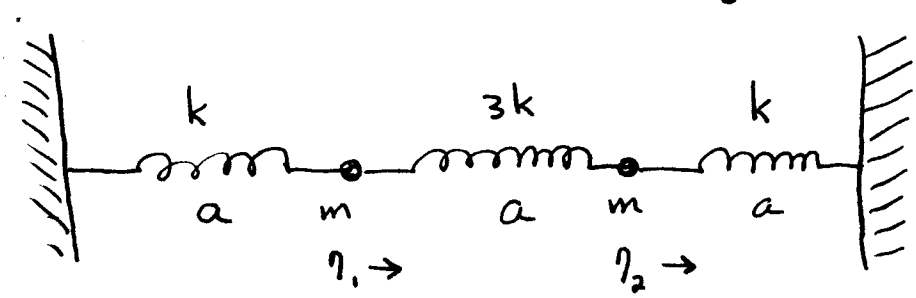
$$\eta_3 = \frac{1}{\omega_3} a_{23} M v_0 a_{33} e^{-i(\omega_3 t - \pi/2)}$$

Actual Motion: $\eta_i = \eta_i^{amp} \cos(\omega_3 t - \pi/2)$
(real)

Also: $\xi_2 = C_2 e^{-i\omega_2 t} = 0 \qquad \xi_3 = C_3 e^{-i\omega_3 t}$

$$= \frac{1}{\omega_3} a_{23} M v_0 e^{-i(\omega_3 t - \pi/2)}$$

E 12



$$T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) \qquad V = \frac{1}{2} k (x_1 - a)^2 + \frac{3k}{2}(x_2 - x_1 - a)^2 + \frac{1}{2} k (2a - x_2)^2$$

$$\eta_1 = x_1 - a \quad \eta_2 = x_2 - 2a \Rightarrow V = \frac{1}{2} k \eta_1^2 + \frac{3k}{2}(\eta_2 - \eta_1)^2 + \frac{1}{2} k \eta_2^2$$

$$T = \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2) \quad V = \frac{1}{2}\left[4k\eta_1^2 + 4k\eta_2^2 - 3k\eta_1\eta_2 - 3k\eta_2\eta_1\right]$$

$$\underset{\approx}{V} = \begin{bmatrix} 4k & -3k \\ -3k & 4k \end{bmatrix} \qquad \underset{\approx}{T} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$|V - \lambda T| = 0 \Rightarrow \begin{vmatrix} 4k - m\lambda & -3k \\ -3k & 4k - m\lambda \end{vmatrix} = 0$$

$$(4k - \lambda m)^2 = 9k^2 \qquad 4k - \lambda m = \pm 3k$$

$$\lambda = \frac{1}{m}\{4k \mp 3k\} \qquad \lambda_1 = \frac{k}{m} \qquad \lambda_2 = \frac{7k}{m}$$

$$\omega_1 = \sqrt{k/m} \qquad \omega_2 = \sqrt{\frac{7k}{m}}$$

<u>Eigenvectors</u>

$$(V - \lambda_i T)a = 0 \qquad (V_{11} - \lambda_i T_{11})a_{1i} + V_{12} a_{2i} = 0$$

$$V_{12} a_{1i} + (V_{22} - \lambda_i T_{22}) a_{2i} = 0$$

$$\lambda_1 = \frac{k}{m} \qquad (4k - m\lambda_i)a_{1i} - 3k a_{2i} = 0$$

$$\lambda_2 = \frac{7k}{m} \qquad -3k a_{1i} + (4k - m\lambda_i)a_{2i} = 0$$

$$m\lambda_1 = k \quad \therefore \quad 3k a_{11} - 3k a_{21} = 0 \Rightarrow a_{21} = a_{11}$$

$$m\lambda_2 = 7k \quad \therefore \quad -3k a_{12} - 3k a_{22} = 0 \Rightarrow a_{22} = -a_{12}$$

<u>Mass Normalization</u>

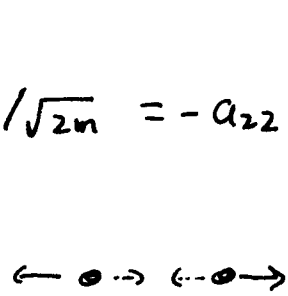
$$\tilde{a}\, T\, a = 1 = [a_{1i} \;\; a_{2i}] \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} a_{1i} \\ a_{2i} \end{bmatrix} = m(a_{1i}^2 + a_{2i}^2)$$

$\underline{a_1}$ $\quad 1 = m(a_{11}^2 + a_{11}^2) = 2m a_{11}^2 \qquad a_{11} = 1/\sqrt{2m} = a_{21}$

$$a_1 = \begin{bmatrix} 1/\sqrt{2m} \\ 1/\sqrt{2m} \end{bmatrix}$$

$\underline{a_2}$ $\quad 1 = m(a_{12}^2 + a_{22}^2) = 2m a_{12}^2 \quad a_{12} = 1/\sqrt{2m} = -a_{22}$

$$a_2 = \begin{bmatrix} 1/\sqrt{2m} \\ -1/\sqrt{2m} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1/\sqrt{2m} \\ 1/\sqrt{2m} \end{bmatrix}$$

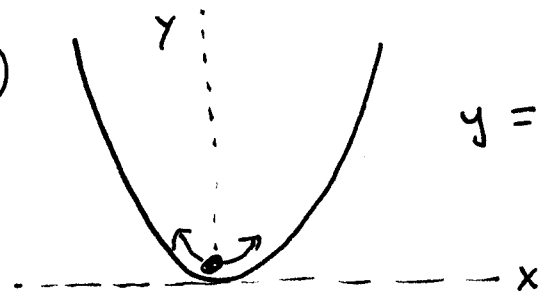


Normal Coordinates

$$\eta_1 = C_1 e^{-i\omega_1 t} \qquad \eta_2 = C_2 e^{-i\omega_2 t}$$

C_1, C_2 determined by initial conditions.

E 16



$y = ax^4 \qquad V = mgy = mgax^4$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ $\quad$ Choose x as generalized coordinate.

$\dot{y} = 4ax^3\dot{x}$

$$T = \frac{m}{2}(\dot{x}^2 + 16a^2x^6\dot{x}^2) = \frac{m}{2}\dot{x}^2(1 + 16a^2x^6)$$

$$L = T - V = \frac{m}{2}\dot{x}^2(1 + 16a^2x^6) - mgax^4$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\left[m\dot{x}(1 + 16a^2x^6)\right]$$

$$- \frac{m}{2}\dot{x}^2(96a^2x^5) + 4mgax^3$$

$$0 = m\ddot{x}(1 + 16a^2x^6) + m\dot{x}\left[96a^2x^5\dot{x}\right] - 48ma^2x^5\dot{x}^2 + 4mgax^3$$

$$0 = m\ddot{x}(1 + 16a^2x^6) + 48ma^2x^5\dot{x}^2 + 4mgax^3$$

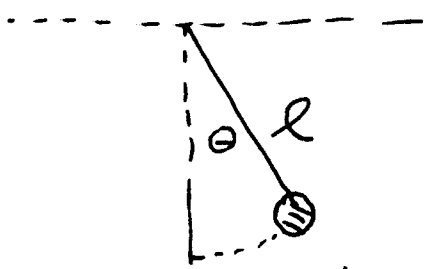
$x \to 0$ small oscillations

$$0 = m\ddot{x} + \underbrace{48ma^2x^5\dot{x}^2}_{\text{if this term is the smaller, then:}} + 4mgax^3$$

$$0 = m\ddot{x} + 4mgax^3 \qquad \boxed{\text{Clearly not SHM}}$$

<u>Problems With Hamilton's Equations</u>

(i) spherical pendulum



generalized coordinates: θ, φ

$$V = -mg\ell\cos\theta$$

$$T = \frac{m}{2}\left[\ell^2\dot{\theta}^2 + \ell^2\sin^2\theta\,\dot{\varphi}^2\right]$$

φ : azimuthal angle

$$\mathcal{H} = T + V \qquad \mathcal{H} = \frac{1}{2}\tilde{p}\,\underset{\approx}{T}^{-1}p + V$$

$$\underset{\approx}{T} = \begin{bmatrix} m\ell^2 & 0 \\ 0 & m\ell^2\sin^2\theta \end{bmatrix} \Rightarrow \underset{\approx}{T}^{-1} = \begin{bmatrix} 1/m\ell^2 & 0 \\ 0 & 1/m\ell^2\sin^2\theta \end{bmatrix}$$

$$\mathcal{H} = \frac{1}{2}\frac{p_\theta^2}{m\ell^2} + \frac{1}{2}\frac{p_\varphi^2}{m\ell^2\sin^2\theta} - mg\ell\cos\theta$$

φ : cyclic coordinate p_φ constant of motion

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \qquad \dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{p_\varphi^2}{m\ell^2}\sin^{-3}\theta\cos\theta - mg\ell\sin\theta$$

$$(p_\theta = m\ell^2\dot{\theta})$$

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial p_\varphi} = \frac{p_\varphi}{m\ell^2\sin^2\theta} \qquad \dot{p}_\varphi = -\frac{\partial \mathcal{H}}{\partial \varphi} = 0 \quad p_\varphi \text{ conserved. } (L_z)$$

$$(p_\varphi = m\ell^2\sin^2\theta\,\dot{\varphi})$$

(ii) From Goldstein (5.52) or class notes:

$$L = \underbrace{\frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2}_{T} \underbrace{- M_g\ell\cos\theta}_{-V}$$

$$\mathcal{H} = T + V$$

$\underset{\approx}{T}$ matrix not diagonal because of $\dot{\psi}\,\dot{\varphi}$ cross term, so use basic method.

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (I_1\sin^2\theta + I_3\cos^2\theta)\,\dot{\varphi} + I_3\cos\theta\,\dot{\psi}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3\{\dot{\psi} + \dot{\varphi}\cos\theta\}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_1\dot{\theta}$$

Cyclic coordinates: φ, ψ $\Rightarrow$ p_φ p_ψ constants of motion

$$\mathcal{H} = T + V = \frac{I_1}{2}(\underbrace{\dot{\theta}^2}_{(p_\theta/I_1)^2} + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}\underbrace{(\dot{\psi} + \dot{\varphi}\cos\theta)^2}_{(p_\psi/I_3)^2} + M_g\ell\cos\theta$$

$$\mathcal{H} = \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{I_1}{2}\sin^2\theta\,\dot{\varphi}^2 + M_g\ell\cos\theta$$

Must write $\dot{\varphi}^2$ in terms of momenta!

$$P_\varphi = (I_1 \sin^2\theta + I_3 \cos^2\theta)\,\dot{\varphi} + I_3 \cos\theta\,\dot{\psi}$$

$$\cos\theta\, P_\psi = I_3 \cos^2\theta\,\dot{\varphi} + I_3\,\dot{\psi}\cos\theta$$

$$P_\varphi - \cos\theta\, P_\psi = I_1 \sin^2\theta\,\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{P_\varphi - \cos\theta\, P_\psi}{I_1 \sin^2\theta}$$

$$\mathcal{H} = \frac{P_\theta^2}{2I_1} + \frac{P_\psi^2}{2I_3} + \frac{1}{2I_1 \sin^2\theta}(P_\varphi - \cos\theta P_\psi)^2 + Mg\ell\cos\theta$$

φ: $$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial P_\varphi} = \frac{1}{2I_1 \sin^2\theta}\, 2(P_\varphi - \cos\theta P_\psi) = \frac{P_\varphi - \cos\theta P_\psi}{I_1 \sin^2\theta}$$ rate of precession

$$\dot{P}_\varphi = -\frac{\partial \mathcal{H}}{\partial \varphi} = 0$$ P_φ conserved

ψ: $$\dot{\psi} = \frac{\partial \mathcal{H}}{\partial P_\psi} = \frac{P_\psi}{I_3} + \frac{(P_\varphi - \cos\theta P_\psi)(-\cos\theta)}{I_1 \sin^2\theta}$$ spinning rate

$$\dot{P}_\psi = -\frac{\partial \mathcal{H}}{\partial \psi} = 0$$ P_ψ conserved

θ: $$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial P_\theta} = \frac{P_\theta}{I_1}$$ nutation rate

$$\dot{P}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{-1}{2I_1 \sin^2\theta}\, 2(P_\varphi - \cos\theta P_\psi)\sin\theta P_\psi + \frac{(P_\varphi - \cos\theta P_\psi)^2}{2I_1}\left[+2\sin^{-3}\theta\cos\theta\right] + Mg\ell\sin\theta$$

or $$\dot{P}_\theta = -\frac{(P_\varphi - \cos\theta P_\psi)P_\psi}{I_1 \sin\theta} + \frac{(P_\varphi - \cos\theta P_\psi)^2}{I_1 \sin^3\theta}\cos\theta + Mg\ell\sin\theta$$

(iii) $V = \frac{1}{2} k r^2$ central force motion $V = \frac{1}{2} k(x^2 + y^2)$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \qquad L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\mathcal{H} = T + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} k x^2 + \frac{1}{2} k y^2$$

$$x: \quad \dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

$$y: \quad \dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \qquad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = -ky$$

$$\text{or} \quad \ddot{x} = \dot{p}_x/m = -k/m\, x \qquad \ddot{y} = \dot{p}_y/m = -k/m\, y$$

(iv) $$L = m_0 c^2 \left\{ 1 - \left[1 - \frac{\dot{x}^2}{c^2} \right]^{1/2} \right\} - V(x)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = -m_0 c^2 \frac{1}{2} \left[1 - \frac{\dot{x}^2}{c^2} \right]^{-1/2} \left(-\frac{2\dot{x}}{c^2} \right)$$

$$p_x = \frac{m_0 c^2 \dot{x}}{c^2} \frac{1}{\sqrt{1 - \dot{x}^2/c^2}} = \frac{m_0 \dot{x}}{\sqrt{1 - \dot{x}^2/c^2}}$$

$$p_x^2 = \frac{m_0^2 \dot{x}^2}{1 - \dot{x}^2/c^2} \Rightarrow p_x^2 - \frac{p_x^2 \dot{x}^2}{c^2} = m_0^2 \dot{x}^2$$

$$\dot{x}^2 \left\{ m_0^2 + p_x^2/c^2 \right\} = p_x^2 \qquad \dot{x}^2 = \frac{p_x^2}{m_0^2 + p_x^2/c^2}$$

$$1 - \frac{\dot{x}^2}{c^2} = 1 - \frac{p_x^2/c^2}{m_0^2 + p_x^2/c^2} = \frac{m_0^2}{m_0^2 + p_x^2/c^2}$$

$$\cancel{\mathcal{H} = T + V(x) = m_0 c^2 \left[1 - \frac{m_0}{\sqrt{m_0^2 + p_x^2/c^2}} \right] + V(x)}$$

$$\mathcal{H} = \dot{q}_i p_i - L = \dot{x} p_x - L$$

$$\mathcal{H} = \frac{p_x^2}{\sqrt{m_0^2 + p_x^2/c^2}} - m_0 c^2 + m_0 c^2 \left[1 - \frac{\dot{x}^2}{c^2} \right]^{1/2} + V(x)$$

$$\mathcal{H} = \frac{p_x^2}{\sqrt{m_0^2 + p_x^2/c^2}} - m_0 c^2 + m_0 c^2 \left[\frac{m_0^2}{m_0^2 + p_x^2/c^2} \right]^{1/2} + V(x)$$

$$\mathcal{H} = \frac{p_x^2 + m_0^2 c^2}{\sqrt{m_0^2 + p_x^2/c^2}} - m_0 c^2 + V(x)$$

$$\mathcal{H} = c^2 \sqrt{p_x^2/c^2 + m_0^2} - m_0 c^2 + V(x)$$

$$\boxed{\mathcal{H} = m_0 c^2 \sqrt{(p_x/m_0 c)^2 + 1} - m_0 c^2 + V(x)}$$

$$x: \quad \dot{x} = \frac{\partial \mathcal{H}}{\partial \dot{x}} = m_0 c^2 \, \frac{1}{2} \left[1 + (p_x/m_0 c)^2 \right]^{-1/2} \frac{2 p_x}{m_0^2 c^2}$$

$$\dot{x} = \frac{p_x/m_0}{\sqrt{1 + (p_x/m_0 c)^2}}$$

$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial V}{\partial x}$$