Physics 821
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Answers To Set # 5

Goldstein Chap 4

(D3) $\chi(BAB^{-1}) = \sum_i (BAB^{-1})_{ii}$

a) or $\chi = b_{ij}\, a_{jk}\, b^{-1}{}_{ki} = b^{-1}{}_{ki}\, b_{ij}\, a_{jk}$

$\chi = (1)_{kj}\, a_{jk} = \delta_{kj}\, a_{jk} = a_{kk}$

b) $\tilde{B} = B^{-1}$ $\quad a_{ji} = -a_{ij}$

Consider $(BA\tilde{B})_{ji}$ $\quad b_{jl}\, a_{lk}\, \tilde{b}_{ki} = b_{jl}\, a_{lk}\, b_{ik}$

$= -b_{jl}\, a_{kl}\, b_{ik} = -b_{ik}\, a_{kl}\, b_{jl}$

$= -b_{ik}\, a_{kl}\, \tilde{b}_{lj} = -(BA\tilde{B})_{ij}$

(D6) First, rotate around x axis by θ $\quad \xi = CX;\; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

Secondly, rotate about ξ axis by ψ $\quad \xi' = B\xi \quad B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xi' = BCX$

Now $D = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rotates $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ around z axis by φ

The matrix that rotates BCX around old z axis by φ is

$(BC)D(BC)^{-1}$ [similarity transformation]

$\therefore$ Total rotation $= \underbrace{(BC)D(BC)^{-1}BC}_{BCDC^{-1}B^{-1}BC \;=\; BCD \;=\; A} X$

D15 <u>Method 1</u>:

$$\omega_x = (\omega_\varphi)_x + (\omega_\theta)_x + (\omega_\psi)_x$$
$$\omega_y = (\omega_\varphi)_y + (\omega_\theta)_y + (\omega_\psi)_y$$
$$\omega_z = (\omega_\varphi)_z + (\omega_\theta)_z + (\omega_\psi)_z$$

$\underline{\vec{\omega}_\varphi}$ around z $\therefore$

$$\begin{bmatrix} (\omega_\varphi)_x \\ (\omega_\varphi)_y \\ (\omega_\varphi)_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix}$$

$\underline{\vec{\omega}_\theta}$ around line of nodes

$$\begin{bmatrix} (\omega_\theta)_x \\ (\omega_\theta)_y \\ (\omega_\theta)_z \end{bmatrix} = \underbrace{D^{-1}}_{\tilde{D}} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\vec{\omega}_\psi}$ around z'

$$\begin{bmatrix} (\omega_\psi)_x \\ (\omega_\psi)_y \\ (\omega_\psi)_z \end{bmatrix} = \underbrace{A^{-1}}_{\tilde{A}} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\therefore \quad \omega_x = \cos\varphi\,\dot{\theta} + A_{31}\dot{\psi} \underset{\text{p. 153}}{\overset{\text{See}}{=}} \cos\varphi\,\dot{\theta} + \sin\theta\sin\phi\,\dot{\psi}$$

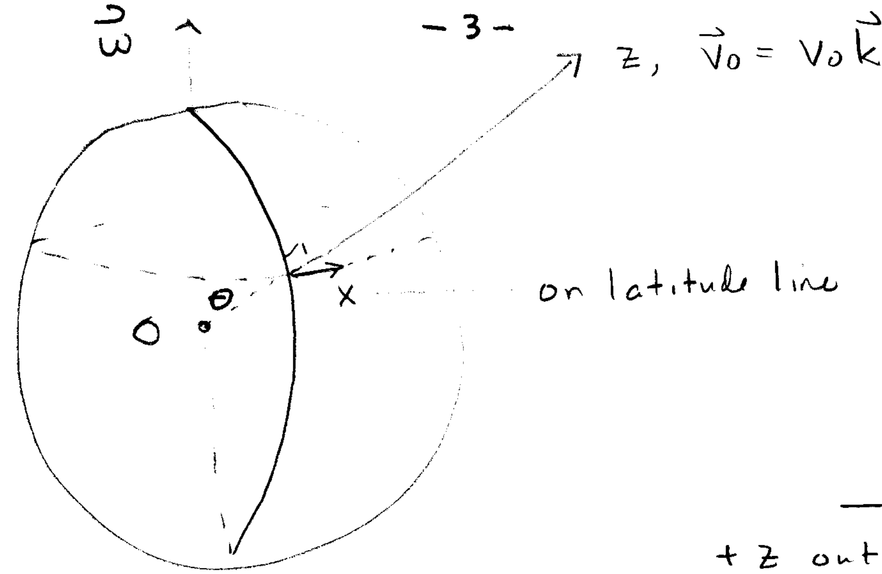
$$\omega_y = \sin\varphi\,\dot{\theta} + A_{32}\dot{\psi} = \sin\varphi\,\dot{\theta} - \sin\theta\cos\varphi\,\dot{\psi}$$

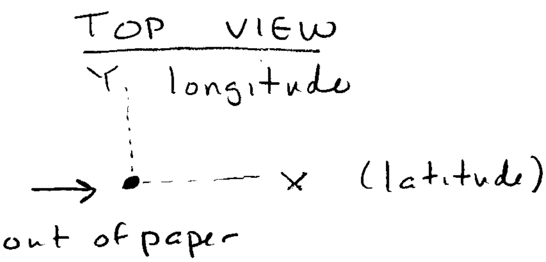
$$\omega_z = \dot{\varphi} + A_{33}\dot{\psi} = \dot{\varphi} + \cos\theta\,\dot{\psi}$$

Method 2: Transform $\omega_{x'}, \omega_{y'}, \omega_{z'}$ derived in class

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = A^{-1} \begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix} = \tilde{A} \begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix}$$

E21





Due to Coriolis Term:

$$m a_x = -2m|(\vec{\omega} \times \vec{v})| \qquad V_z > 0 \quad \text{Force in } -x \text{ direction (clockwise)}$$

$$m\frac{dV_x}{dt} = -2m\omega V_z \sin\theta \qquad V_z < 0 \quad \text{Force in } +x \text{ direction (counterclockwise)}$$

① Particle Goes Up & Down

$$V_z = V_0 - gt \qquad \text{At top: } V_0 = g t_{top}$$

$$\frac{dV_x}{dt} = -2\omega\sin\theta\,(V_0 - gt) = -2\omega V_0 \sin\theta + 2\omega\sin\theta\, g t$$

$$V_x = -2\omega\sin\theta\, V_0 t + \omega\sin\theta\, g t^2 \qquad (V_x(0) = 0)$$

$$x = -\omega\sin\theta\, V_0\, t^2 + \omega g \sin\theta\, t^3/3 \qquad (x(0) = 0)$$

When particle hits ground:

$$t = 2V_0/g \qquad (z = V_0 t - \tfrac{1}{2} g t^2)$$

$$x = -\omega\sin\theta\, V_0 \left[\frac{2V_0}{g}\right]^2 + \omega g \sin\theta \left(\frac{2V_0}{g}\right)^3 \frac{1}{3}$$

$$x = \frac{\omega V_0^3}{g^2}\sin\theta \left[-4 + 8/3\right] = -\frac{4}{3}\omega V_0^3 \sin\theta / g^2$$

② Dropped From Height

$$V_z = -gt \qquad t = V_0/g \text{ to reach ground}$$

$$\frac{dV_x}{dt} = -2\omega\sin\theta\,(-gt) = 2\omega\sin\theta\, g t$$

$$V_x = \omega \sin\theta\, g\, t^2 \qquad x = \tfrac{1}{3}\, \omega \sin\theta\, g\, t^3$$

$$x = \tfrac{1}{3}\, \omega \sin\theta\, V_0^3 / g^2$$

$$\frac{X_{up-down}}{X_{down}} = -4$$

Goldstein Chapter 5

(D2) Consider right-handed set $\hat{\alpha}, \hat{\beta}, \hat{n}$ of unit vectors.

$$\vec{r}_i = r_{i\alpha}\hat{\alpha} + r_{i\beta}\hat{\beta} + r_{in}\hat{n} \qquad r_{in} = \vec{r}_i \cdot \hat{n}$$

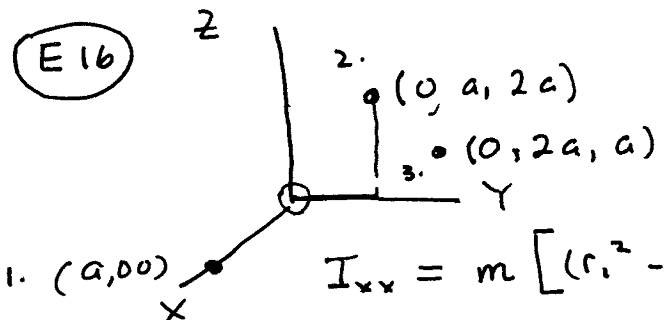
$$\vec{r}_i \times \hat{n} = r_{i\alpha}(\hat{\alpha}\times\hat{n}) + r_{i\beta}(\hat{\beta}\times\hat{n}) \qquad \hat{n}\times\hat{n} = 0$$

$$= -r_{i\alpha}\hat{\beta} + r_{i\beta}\hat{\alpha}$$

$$(\vec{r}_i \times \hat{n})\cdot(\vec{r}_i \times \hat{n}) = r_{i\alpha}^2 + r_{i\beta}^2 = r_i^2 - (\vec{r}_i\cdot\hat{n})^2$$

$$\therefore\ I = m_i(\vec{r}_i\times\hat{n})\cdot(\vec{r}_i\times\hat{n}) = m_i\left[r_i^2 - (\vec{r}_i\cdot\hat{n})^2\right]$$

(E16)



about origin; not C of M

$$I_{xx} = m\left[(r_1^2 - x_1^2) + (r_2^2 - x_2^2) + (r_3^2 - x_3^2)\right]$$

$$= m\left[(a^2 - a^2) + (5a^2) + (5a^2)\right]$$

$$I_{xx} = 10\, m a^2$$

$$I_{yy} = m\left[(r_1^2 - y_1^2) + (r_2^2 - y_2^2) + (r_3^2 - y_3^2)\right]$$

$$= m\left[a^2 + 4a^2 + a^2\right] = 6ma^2$$

$$I_{zz} = m\left[(r_1^2 - z_1^2) + (r_2^2 - z_2^2) + (r_3^2 - z_3^2)\right]$$

$$= m\left[a^2 + a^2 + 4a^2\right] = 6ma^2$$

$$I_{xy} = -m\left[x_1y_1 + x_2y_2 + x_3y_3\right] = 0 = I_{yx}$$

$$I_{xz} = -m\left[x_1 z_1 + x_2 z_2 + x_3 z_3\right] = 0 = I_{zx}$$

$$I_{yz} = -m\left[y_1 z_1 + y_2 z_2 + y_3 z_3\right] = -4ma^2 = I_{zy}$$

$$I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$$

$$\begin{vmatrix} 10-\lambda & 0 & 0 \\ 0 & 6-\lambda & -4 \\ 0 & -4 & 6-\lambda \end{vmatrix} = 0$$

method of minors: $(10-\lambda)\left[(6-\lambda)^2 + -16\right] = 0$

$\lambda = 10$ $\quad (\lambda-6)^2 = 4^2 \quad \lambda - 6 = \pm 4$

$\lambda = 10, 2$

$\therefore \lambda = 10, 10, 2 \qquad I = 10ma^2, 10ma^2, 2ma^2$

prolate symmetric top

<u>Axes</u>

$$\begin{bmatrix} 10ma^2 - I_i & 0 & 0 \\ 0 & 6ma^2 - I_i & -4ma^2 \\ 0 & -4ma^2 & 6ma^2 - I_i \end{bmatrix} \underbrace{\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}}_{R_i} = 0$$

Now:

$I_1 = 2ma^2$ (figure axis)

$8ma^2 X_1 = 0 \qquad 4ma^2 Y_1 - 4ma^2 Z_1 = 0$

$+ \quad -4ma^2 Y_1 + 4ma^2 Z_1 = 0$

$\therefore X_1 = 0 \quad Y_1 = Z_1 \qquad R_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ normalized axis 1 in Y-Z plane slope of unity, through origin

Axis 2 & Axis 3 $\quad I_2 = I_3 = 10ma^2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4ma^2 & -4ma^2 \\ 0 & -4ma^2 & -4ma^2 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = 0$$

$0 \cdot X_2 = 0$ $\quad -4ma^2(Y_2 + Z_2) = 0 \quad Z_2 = -Y_2$

no info

Same for R_3: $\quad 0 \cdot X_3 = 0 \quad Z_3 = -Y_3$

Idea is to have 3 perpendicular axes.

Let's choose X_2 & X_3.

Axis 2

Arbitrarily set $X_2 = 0 \quad \therefore \quad R_2$ in YZ plane $\perp$ to R_1

$$R_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad R_1 \cdot R_2 = 0$$

Axis 3

Choose X axis $\quad$ (Set $Y_3 = 0 \Rightarrow Z_3 = 0$)

$$R_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$