Physics 821 Spring 2007

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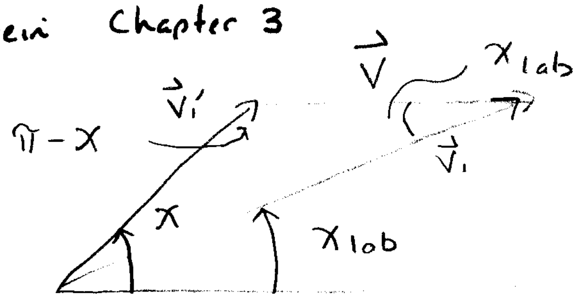
# Answers To Set #4

Goldstein Chapter 3

(8)




$$E_0 = \frac{1}{2} m_1 v_0^2$$

$$E_1 = \frac{1}{2} m_1 v_1^2$$

(energy of particle 1 after collision)

Law of Cosines:

$$v_1^2 = v_1'^2 + V^2 - 2v_1' V \cos(\pi - \chi)$$

$$v_1^2 = v_1'^2 + V^2 + 2v_1' V \cos \chi$$

but $v_1' = \mu/m_2 \, v_0/\rho$ (class notes; Goldstein 3.108)

$$\therefore \; v_1^2 = (\mu/m_2\rho)^2 v_0^2 + (\mu/m_2)^2 v_0^2 + 2\left(\frac{\mu}{m_2}\right)^2 \frac{v_0^2}{\rho} \cos\chi$$

$(V = \mu/m_2 \, v_0)$ $\qquad \mu = m_1 m_2/(m_1+m_2)$

$$\therefore \; \frac{v_1^2}{v_0^2} = (\mu/m_2\rho)^2 \left[1 + 2\rho\cos\chi + \rho^2\right]$$

$$\therefore \; \frac{E_1}{E_0} = \left(\frac{\mu}{m_2\rho}\right)^2 \left[1 + 2\rho\cos\chi + \rho^2\right] \qquad (1)$$

but $\quad v_1' \cos\chi + V = v_1 \cos\chi_{lab}$

$$\cos\chi = \frac{v_1 \cos\chi_{lab} - V}{v_1'} = \frac{v_1}{v_1'} \cos\chi_{lab} - \frac{V}{v_1'}$$

$$\cos\chi = v_1 \rho/V \, \cos\chi_{lab} - \rho$$

since $\rho = \frac{\mu}{m_2} \frac{v_0}{v_1'} = \frac{V}{v_1'}$

Also $\quad E_1/E_0 = v_1^2/v_0^2 = \frac{v_1^2 \mu^2}{m_2^2 V^2}$ ; $\sqrt{\frac{E_1}{E_0}} = \frac{v_1}{V} \frac{\mu}{m_2}$

$$\cos\chi = \sqrt{\frac{E_1}{E_0}}\ \frac{m_2}{\mu}\ p\cos\chi_{lab} - p$$

From (1):

$$\frac{E_1}{E_0} = \left(\frac{\mu}{m_2 p}\right)^2 \left\{1 + 2p^2\left[\sqrt{\frac{E_1}{E_0}}\ \frac{m_2}{\mu}\cos\chi_{lab} - 1\right] + p^2\right\}$$

$$\frac{E_1}{E_0} = \frac{\mu^2}{m_2^2 p^2} + \frac{\mu^2}{m_2^2} + \frac{2\mu}{m_2}\sqrt{\frac{E_1}{E_0}}\cos\chi_{lab} - 2\mu^2/m_2^2$$

$$\frac{E_1}{E_0} = \frac{\mu^2}{m_2^2 p^2} - \frac{\mu^2}{m_2^2} + \frac{2\mu}{m_2}\sqrt{\frac{E_1}{E_0}}\cos\chi_{lab}$$

$$\cos\chi_{lab} = \frac{m_2}{2\mu}\sqrt{\frac{E_0}{E_1}}\left[E_1/E_0 - \frac{\mu^2}{m_2^2}\left(\frac{1}{p^2} - 1\right)\right]$$

$$= \frac{m_1+m_2}{2m_1}\sqrt{\frac{E_1}{E_0}} - \frac{\mu}{2m_2}\left(\frac{1}{p^2} - 1\right)\sqrt{\frac{E_0}{E_1}}$$

$$p^2 = \frac{m_1^2}{m_2^2}\ \frac{1}{1 + (\mu/m_1)^{-1}Q/E_0} \qquad (3.114)$$

$$\therefore\ \cos\chi_{lab} = \frac{m_1+m_2}{2m_1}\sqrt{\frac{E_1}{E_0}} - \frac{\mu}{2m_2}\sqrt{\frac{E_0}{E_1}}\left\{\frac{m_2^2}{m_1^2}\left(1 + \left[\frac{\mu}{m_1}\right]^{-1}\frac{Q}{E_0}\right) - 1\right\}$$

$$\cos\chi_{lab} = \frac{m_1+m_2}{2m_1}\sqrt{\frac{E_1}{E_0}} - \frac{m_1}{2(m_1+m_2)}\sqrt{\frac{E_0}{E_1}}\left\{\frac{m_2^2 - m_1^2}{m_1^2}\right\}$$

$$- \frac{\mu}{2m_2}\ \frac{m_2^2}{m_1^2}\left[\frac{\mu}{m_1}\right]^{-1}\frac{Q}{\sqrt{E_0E_1}}$$

$$\cos\chi_{lab} = \frac{m_1+m_2}{2m_1}\sqrt{\frac{E_1}{E_0}} - \frac{m_2-m_1}{2m_1}\sqrt{\frac{E_0}{E_1}} - \frac{m_2}{2m_1}\ \frac{Q}{\sqrt{E_0E_1}}$$

(31) $F = k/r^3$ repulsive $V(r) = k/2r^2$ $u = 1/r$

$$\sigma(\chi) = \frac{b}{\sin\chi} \left| \frac{db}{d\chi} \right| \qquad \chi = \pi - 2b \int_0^{u_{max}} \frac{du}{\sqrt{1 - ku^2/2E - b^2u^2}}$$

$r_{min} = U_{max}$

$\dot{r} = 0$ $\qquad E = V(r_{min}) + E\, b^2/r_{min}^2 = k/2r_{min}^2 + E\, b^2/r_{min}^2$

$E > 0$

$$\therefore r_{min} = \frac{1}{U_{max}} = \sqrt{b^2 + k/2E}$$

$$\therefore \chi = \pi - 2b \int_0^{U_{max}} \frac{du}{\sqrt{1 - (U/U_{max})^2}} \qquad \text{Let } y = U/U_{max} \quad 0 \le y \le 1$$

$$\chi = \pi - 2b\,U_{max} \underbrace{\int_0^{U_{max}/U_{max}=1} \frac{dy}{\sqrt{1-y^2}}}_{\pi/2}$$

$$\chi = \pi - \pi b \left[ b^2 + k/2E \right]^{-1/2} = \pi \left[ 1 - \frac{b}{\sqrt{b^2 + k/2E}} \right]$$

$$x = \frac{\chi}{\pi} \Rightarrow \quad x - 1 = \frac{-b}{\sqrt{b^2 + k/2E}} \quad \Longrightarrow \quad b^2 = b^2(x-1)^2 + \frac{k}{2E}(x-1)^2$$

$x > 0$

$$b^2 = \frac{k/2E\,(x-1)^2}{1-(x-1)^2} = \frac{k}{2E}\,\frac{(x-1)^2}{2x - x^2}$$

$$2b\frac{db}{dx} = \frac{k}{2E}\left\{ \frac{2(x-1)}{2x - x^2} + (x-1)^2\left[(-1)\left[2x - x^2\right]^{-2}(2-2x)\right] \right\}$$

$$= \frac{k}{2E}\left\{ \frac{2(x-1)}{2x - x^2} - \frac{(x-1)^2\, 2(1-x)}{(2x-x^2)^2} \right\}$$

$$2b\frac{db}{dx} = \frac{k}{2E}\left\{\frac{2(x-1)(2x-x^2) - (x-1)^2\,2\,(1-x)}{(2x-x^2)^2}\right\}$$

$$= \frac{2k}{2E}\left\{\frac{2x-x^2 + (x-1)^2}{(2x-x^2)^2}\right\}(1-x)$$

$$b\frac{db}{dx} = \frac{k}{2E}\left\{\frac{1-x}{(2x-x^2)^2}\right\}$$

Note: $(2x-x^2)^2 = 4x^2 - 4x^3 + x^4 = x^2(4-4x+x^2)$
$= x^2(2-x)^2$

$$\sigma(\chi) = b\frac{db}{dx}\frac{dx}{d\chi}$$

$$\sigma(\chi)\,d\chi = \frac{k}{2E}\,\frac{(1-x)\,dx}{(2-x)^2 x^2 \sin\pi x} \qquad 0 \le x \le 1$$

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Repulsive Scattering


As $x \to 0$ $\quad \sigma(\chi) \to \infty$

As $x \to 1$ $\quad \sigma(\chi)$ finite – use L'Hôpital's rule.

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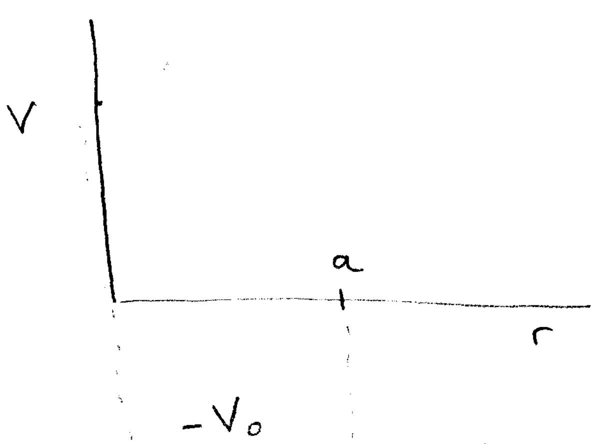
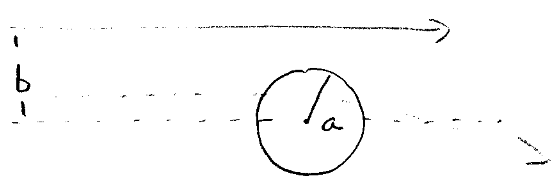



$b > a$ no scattering

$r_{min}$ $(b < a)$

$$E = -V_0 + E\,b^2/r_{min}^2$$

$$r_{min} = b\sqrt{\frac{E}{E+V_0}} = \frac{b}{n} = \frac{1}{U_{max}}$$

$$\chi = \pi - 2b \int_0^{u_{max}} \frac{du}{\sqrt{1 - V(u)/E - b^2u^2}}$$

$$\chi = \pi - 2b \left[ \int_0^{1/a} \frac{du}{\sqrt{1-b^2u^2}} + \int_{1/a}^{n/b} \frac{du}{\sqrt{\underbrace{1 + V_0/E}_{n^2} - b^2u^2}} \right]$$

Let $y = bu$

$$\chi = \pi - 2 \left[ \int_0^{b/a} \frac{dy}{\sqrt{1-y^2}} + \int_{b/a}^{n} \frac{dy}{\sqrt{n^2-y^2}} \right]$$

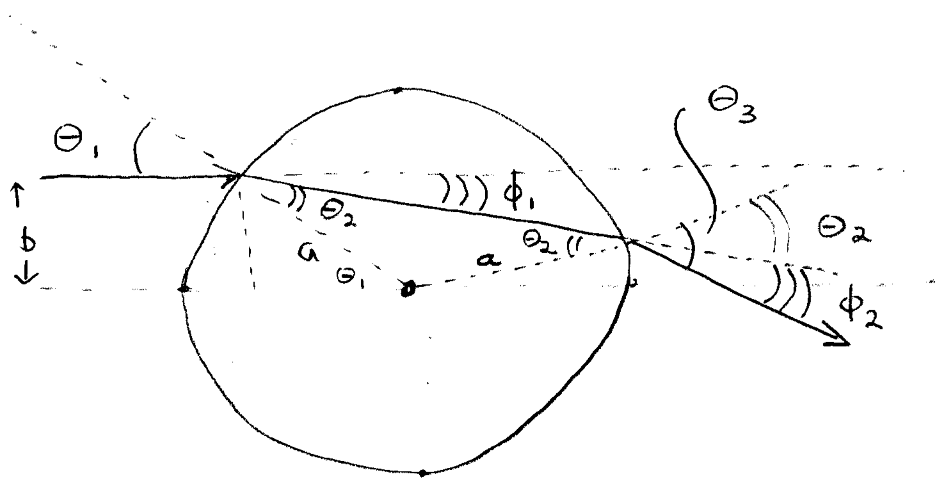
$$\chi = \pi - 2 \left[ \sin^{-1} y \Big|_0^{b/a} + \sin^{-1} y/n \Big|_{b/a}^{n} \right]$$

$$\chi = \pi - 2 \left[ \sin^{-1} b/a - \underbrace{\sin^{-1} 0}_{0} + \underbrace{\sin^{-1} 1}_{\pi/2} - \sin^{-1} b/na \right]$$

$$\chi = \pi - \pi + 2 \left[ \sin^{-1} b/na - \sin^{-1} b/a \right]$$

$$\chi/2 = \sin^{-1} b/na - \sin^{-1} b/a \quad < 0 \quad \underline{\text{attractive}}$$

## Spherical Refraction




$\Theta_1 = \Theta_3$

$\sin\Theta_1 = b/a$

$$\frac{\sin\Theta_1}{\sin\Theta_2} = n = \frac{\sin\Theta_3}{\sin\Theta_2}$$

$n \sin\Theta_2 = b/a$

$\sin\Theta_2 = b/na$

$\chi = \phi_1 + \phi_2 \qquad \phi_1 = \Theta_1 - \Theta_2$

$\phi_2 = \Theta_3 - \Theta_2$

$$\chi = \varphi_1 + \varphi_2 = 2\varphi_1 = 2(\theta_1 - \theta_2)$$

$$\frac{\chi}{2} = \theta_1 - \theta_2 = \sin^{-1} b/a - \sin^{-1} b/na$$

Differential Cross Section $\quad \sigma(\chi) = \dfrac{b}{\sin\chi}\left|\dfrac{db}{d\chi}\right|$

$$\sin\chi = 2\sin\frac{\chi}{2}\cos\frac{\chi}{2}$$

$$\frac{d\chi}{db} = \frac{2}{a}\frac{1}{\sqrt{1-b^2/a^2}} - \frac{2}{na}\frac{1}{\sqrt{1-(b/na)^2}}$$

$$\frac{d\chi}{db} = \frac{2}{a}\frac{1}{\cos\theta_1} - \frac{2}{na}\frac{1}{\cos\theta_2}$$

$$\frac{d\chi}{db} = \frac{2}{na}\left[\frac{n}{\cos\theta_1} - \frac{1}{\cos\theta_2}\right] = \frac{2}{na}\,\frac{n\cos\theta_2 - \cos\theta_1}{\cos\theta_1\cos\theta_2}$$

$$\sigma(\chi) = \frac{b}{2\sin\frac{\chi}{2}\cos\chi/2}\;\frac{na}{2}\;\frac{\cos\theta_1\cos\theta_2}{n\cos\theta_2 - \cos\theta_1}$$

$$\frac{\chi}{2} = \varphi_1 = \theta_1 - \theta_2 \qquad \sin\frac{\chi}{2} = \sin(\theta_1 - \theta_2) = \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1$$

$$\sin\frac{\chi}{2} = \frac{b}{a}\cos\theta_2 - \frac{b}{na}\cos\theta_1 = \frac{b}{na}\left(n\cos\theta_2 - \cos\theta_1\right)$$

$$\therefore\ \sigma(\chi) = \frac{n^2a^2}{4\cos\frac{\chi}{2}}\;\frac{\cos\theta_1\cos\theta_2}{(n\cos\theta_2 - \cos\theta_1)^2}$$

Denominator $= ?$ $\quad$ Note: $\cos\frac{\chi}{2} = \cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$

$$(n\cos\theta_2 - \cos\theta_1)^2 = n^2\cos^2\theta_2 + \cos^2\theta_1 - 2n\cos\theta_1\cos\theta_2$$

$$= n^2\cos^2\theta_2 + \cos^2\theta_1 - 2n\left[\cos\frac{\chi}{2} - \sin\theta_1\sin\theta_2\right]$$

$$= n^2\cos^2\theta_2 + \cos^2\theta_1 + 2n\sin\theta_1\underset{\uparrow\,\sin}{\theta_2} - 2n\cos\chi/2$$

$$1^{ST}\ 3\ \text{terms} = n^2\left[1 - \frac{b^2}{n^2a^2}\right] + 1 - \frac{b^2}{a^2} + 2\frac{n\,b^2}{na^2} = 1 + n^2$$

$$\sigma(x) = \frac{n^2 a^2}{4\cos x/2} \; \frac{\cos\theta_1 \cos\theta_2}{1 + n^2 - 2n\cos x/2}$$

$$= \frac{n^2 a^2}{4\cos x/2} \left[ \frac{\cos x/2 - b^2/na^2}{(1 + n^2 - 2n\cos x/2)^2} \right] \cdot (1 + n^2 - 2n\cos x/2)$$

$$\text{Numerator} = (n^2+1)\cos\frac{x}{2} - n - n\cos^2\frac{x}{2} + n - n\cos^2\frac{x}{2}$$

$$- b^2/na^2 \left[ n^2 + 1 - 2n\cos x/2 \right]$$

$$\text{Numerator} = \left[ \left(n\cos\frac{x}{2} - 1\right)\left(n - \cos\frac{x}{2}\right) \right] + n\sin^2\frac{x}{2}$$

$$- \frac{b^2}{na^2}\left(1 + n^2 - 2n\cos x/2\right)$$

Must show that

$$n\sin^2 x/2 - \frac{b^2}{na^2}\left(1 + n^2 - 2n\cos x/2\right) = 0$$

Consider:

$$n\sin^2\frac{x}{2} = n\sin^2(\theta_1 - \theta_2) = \frac{b^2}{na^2}\left(n\cos\theta_2 - \cos\theta_1\right)^2$$

$$\underset{\text{(see page 6)}}{=} \frac{b^2}{na^2}\left[1 + n^2 - 2n\cos x/2\right]$$

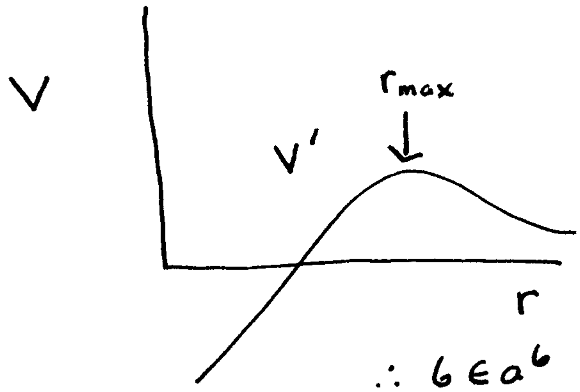
$$\therefore \quad \sigma(x) = \frac{n^2 a^2}{4\cos x/2} \; \frac{\left(n\cos\frac{x}{2} - 1\right)\left(n - \cos x/2\right)}{\left[1 + n^2 - 2n\cos x/2\right]^2}$$

<u>Total Cross Section</u>

$$\sigma_T = \int_0^{b_{max}} 2\pi b\, db = \pi b_{max}^2 \quad ; \quad b_{max} = a$$

$$\therefore \quad \sigma_T = \pi a^2$$

## Lennard - Jones Problem




$$V' = -\frac{\epsilon a^6}{r^6} + \frac{E b^2}{r^2}$$

$$\frac{dV'}{dr} = 0 = \frac{6\epsilon a^6}{r^7} - \frac{2Eb^2}{r^3} = 0$$

$$\therefore 6\epsilon a^6 = 2Eb^2 r^4$$

$$r_{max}^4 = 3\epsilon a^6 / E b^2$$

$$V'(r_{max}) = -\epsilon a^6 \left(\frac{Eb^2}{3\epsilon a^6}\right)^{1.5} + Eb^2\left(\frac{Eb^2}{3\epsilon a^6}\right)^{1/2}$$

$$V'(r_{max}) = \frac{(Eb^2)^{1.5}}{(\epsilon a^6)^{0.5}}\left[-\frac{1}{3^{1.5}} + \frac{1}{3^{0.5}}\right]$$

$$V'(r_{max}) = \frac{(Eb^2)^{1.5}}{(3\epsilon a^6)^{0.5}}\left[1 - \frac{1}{3}\right] = \frac{2}{3}\frac{(Eb^2)^{1.5}}{(3\epsilon a^6)^{0.5}}$$

Orbiting Condition

$$E = V'(r_{max}) \quad \frac{3E}{2} = \frac{(E b_m^2)^{1.5}}{(3\epsilon a^6)^{0.5}}$$

$$\frac{3E}{2}(E)^{-1.5}(3\epsilon a^6)^{0.5} = b_m^3$$

$$b_m^3 = \frac{3}{2}\frac{1}{\sqrt{E}}\sqrt{3\epsilon}\ a^3$$

$$\sigma_T = \pi b_m^2 = \left(\frac{3}{2}\right)^{2/3}\pi a^2 (3\epsilon)^{1/3} E^{-1/3} = A/E^{1/3}$$

Thermal Averaging

$$f_E\, dE = 2\sqrt{\frac{E}{\pi (kT)^3}}\, e^{-E/kT}\, dE \quad \text{in terms of energy}$$

$$k = \langle v\sigma \rangle = \int_0^\infty \left(\frac{2E}{m}\right)^{1/2} A E^{-1/3}\, 2\left(\frac{E}{\pi (kT)^3}\right)^{1/2} e^{-E/kT}\, dE$$

$E = \frac{1}{2} m v^2 \quad v = \left(\frac{2E}{m}\right)^{1/2}$

$$k = \left(\frac{2}{m}\right)^{1/2} A\, 2\left(\frac{1}{\pi (kT)^3}\right)^{1/2} \underbrace{\int_0^\infty E^{2/3} e^{-E/kT}\, dE}_{\Gamma(5/3)(kT)^{+5/3}}$$

$$k = \left(\frac{2}{m}\right)^{1/2} 2 \cdot A\left(\frac{1}{\sqrt{\pi}}\right) \Gamma(5/3)(kT)^{1/6}$$

$k \propto T^{1/6}$ weak temperature dependence

Goldstein Ch. 4

(5) $A = BCD = B \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = B \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\cos\theta\sin\varphi & \cos\theta\cos\varphi & \sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\cos\theta\sin\varphi & \cos\theta\cos\varphi & \sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & \cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}$$

Orthonormality : $a_{ij} a_{ik} = \delta_{jk}$ (Einstein convention)

Columns 1 + 2 ($j=1$ $k=2$)

$(\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)(\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)$

$+ (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)(-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)$

$+ - \sin^2\theta\sin\varphi\cos\varphi$

$= \cos^2\psi\cos\varphi\sin\varphi + \cos\psi\sin\psi\cos^2\varphi\cos\theta - \sin\psi\cos\psi\cos\theta\sin^2\varphi$
$- \sin^2\psi\cos^2\theta\sin\varphi\cos\varphi$

$+ \sin^2\psi\cos\varphi\sin\varphi - \sin\psi\cos\psi\cos^2\varphi\cos\theta + \sin\psi\cos\psi\cos\theta\sin^2\varphi$
$- \cos^2\psi\cos^2\theta\sin\varphi\cos\varphi$

$- \sin^2\theta\sin\varphi\cos\varphi$

$= \cos\varphi\sin\varphi - \cos^2\theta\sin\varphi\cos\varphi - \sin^2\theta\sin\varphi\cos\varphi = 0$ ✓

Columns 1 + 3 ($j=1$ $k=3$)

$(\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)\sin\psi\sin\theta$

$+ (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)\cos\psi\sin\theta + \sin\theta\cos\theta\sin\varphi$

$= \cos\psi\sin\psi\cos\varphi\sin\theta - \sin^2\psi\cos\theta\sin\theta\sin\varphi$
$- \cos\psi\sin\psi\cos\varphi\sin\theta - \cos^2\psi\cos\theta\sin\theta\sin\varphi + \sin\theta\cos\theta\sin\varphi$

$= -\cos\theta\sin\theta\sin\varphi + \cos\theta\sin\theta\sin\varphi \overset{\checkmark}{=} 0$

Columns 2 + 3 ($j=2$ $k=3$)

$(\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)\sin\psi\sin\theta$

$+ (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)\cos\psi\sin\theta - \sin\theta\cos\theta\cos\varphi$

$= \cos\psi\sin\psi\sin\theta\sin\varphi + \sin^2\psi\cos\theta\sin\theta\cos\varphi$
$- \cos\psi\sin\psi\sin\theta\sin\varphi + \cos^2\psi\cos\theta\sin\theta\cos\varphi - \sin\theta\cos\theta\cos\varphi$

$\overset{\checkmark}{=} 0$

## Normalization

Column 1 :

$$(\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)^2 + (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)^2 + \sin^2\theta\sin^2\varphi$$

$$= \cos^2\psi\cos^2\varphi - 2\cos\psi\sin\psi\cos\varphi\cos\theta\sin\varphi + \sin^2\psi\cos^2\theta\sin^2\varphi$$
$$+ \sin^2\psi\cos^2\varphi + 2\cos\psi\sin\psi\cos\varphi\cos\theta\sin\varphi + \cos^2\psi\cos^2\theta\sin^2\varphi$$
$$+ \sin^2\theta\sin^2\varphi = \cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi$$
$$= \cos^2\varphi + \sin^2\varphi = 1 \checkmark$$

Column 2

$$(\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)^2 + (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)^2 + \sin^2\theta\cos^2\varphi$$

$$= \cos^2\psi\sin^2\varphi + 2\cos\psi\sin\psi\sin\varphi\cos\theta\cos\varphi + \sin^2\psi\cos^2\theta\cos^2\varphi$$
$$+ \sin^2\psi\sin^2\varphi - 2\cos\psi\sin\psi\sin\varphi\cos\theta\cos\varphi + \cos^2\psi\cos^2\theta\cos^2\varphi$$
$$+ \sin^2\theta\cos^2\varphi = \sin^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\theta\cos^2\varphi = 1 \checkmark$$

Column 3

$$\sin^2\psi\sin^2\theta + \cos^2\psi\sin^2\theta + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1 \checkmark$$