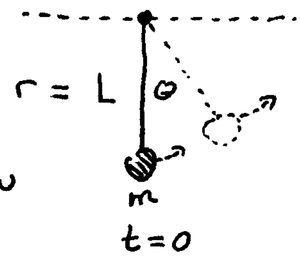
Physics 821 Spring 2007

Answers To Set #3

Chap. 2 (26)



$\dot{\theta}(0) = \omega$

$V = -mgr\cos\theta$

Constraint : $r - L = f = 0$

$\frac{\partial f}{\partial r} = 1 \qquad Q_r = -\lambda_r$

$$L = \tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta \qquad \dot{r} = 0$$

θ eq. : $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgr\sin\theta$

$$\therefore \quad mr^2\ddot{\theta} + mgr\sin\theta = 0 \quad \text{or} \quad \ddot{\theta} + g/L \sin\theta = 0$$

$$\dot{\theta}\ddot{\theta} + g/L\ \dot{\theta}\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{L}\cos\theta\right) = 0$$

$$\therefore \quad \frac{1}{2}\dot{\theta}^2 - \frac{g}{L}\cos\theta = K$$

$$\left\{\begin{matrix}\dot{\theta}(0) = \omega \\ \theta(0) = 0\end{matrix}\right\} \Rightarrow \quad \tfrac{1}{2}\omega^2 - g/L = K$$

$$\therefore \quad \tfrac{1}{2}\dot{\theta}^2 - \frac{g}{L}\cos\theta = \tfrac{1}{2}\omega^2 - g/L$$

$$\dot{\theta}^2 = \omega^2 + 2g/L\,(\cos\theta - 1)$$

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r eq. : $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad -\frac{\partial L}{\partial r} = Q_r = -\lambda_r$

$$+\lambda_r = \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta$$

$$\lambda_r = mL\omega^2 + 2mg(\cos\theta - 1) + mg\cos\theta$$

$$\lambda_r = mL\omega^2 + 3mg\cos\theta - 2mg = 0 \text{ for slacknes}$$

$$\cos\theta_s = \frac{2}{3} - \frac{L\omega^2}{3g} = \frac{1}{3}\left[2 - L\omega^2/g\right]$$

or $\omega^2 = g/L\,(2 - 3\cos\theta_s)$

Now look again at $T_r = 0 = mr\dot{\theta}\big|_s^2 + mg\cos\theta_s$

$$\cos\theta_s = -\,L\dot{\theta}^2/g \le 0$$

$$\therefore\ \theta_s \geq \pi/2 \qquad \cos\theta_s = 0$$

$$\omega^2_{min} = 2g/L$$

Also: $\theta_s \le 3\pi/2$ but this angle cannot be reached when initial motion is to the right.

Maximum ω corresponds to $\theta_s = \pi$ $\cos\theta_s = -1$

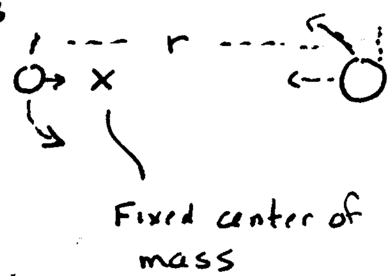
$$\omega^2_{max} = \frac{g}{L}(2+3) = 5g/L$$

$$\therefore\ \omega^2_{min} = 2g/L \qquad \theta_s = \pi/2$$

$$\omega^2_{max} = 5g/L \qquad \theta_s = \pi$$

Chapter 3

(11)



$E = \frac{1}{2}\mu\dot{r}^2 - \frac{k}{r}$ no T_θ

Period: τ

$$\frac{1}{2}\mu\dot{r}^2 = E + k/r$$

$t=0$ $r = r_0$ $\quad E = -k/r_0$ since $\dot{r}_0 = 0$ + E conserved

$\dot{r}_0 = 0$

$$\therefore\ \frac{1}{2}\mu\dot{r}^2 = k/r - k/r_0$$

$$\dot{r} = \sqrt{2k/\mu\left(\frac{1}{r} - \frac{1}{r_0}\right)} = \frac{dr}{dt}$$

$$dt = dr\sqrt{\mu/2k}\,(r_0)^{1/2}\sqrt{r/(r_0 - r)} \qquad dr < 0$$

$$t = \sqrt{\frac{\mu r_0}{2k}} \underbrace{\int_0^{r_0} \sqrt{\frac{r}{r_0 - r}}\, dr}_{\pi r_0/2 \text{ according to Mathematica}}$$

$$t = \frac{\pi}{2} (\mu/2k)^{1/2} (r_0)^{3/2}$$

<u>Circular Motion</u>

$$r_0 = \ell^2/\mu k$$

$$\tau = 2\pi r_0^{3/2} (\mu/k)^{1/2} \quad (\text{p. 101 of Goldstein } a \to r_0)$$

Derivation: $$\tau = \frac{\ell^3}{\mu k^2} \int_0^{2\pi} \frac{d\theta}{(1 + e\cos\theta)^2} \qquad \begin{matrix}(3.66)\\(e=0)\end{matrix}$$

$$\tau_0 = \frac{\ell^3}{\mu k^2} 2\pi = 2\pi r_0^{3/2} (\mu/k)^{1/2}$$

$$t/\tau_0 = \frac{\pi/2}{2\pi} \frac{1}{\sqrt{2}} = 1/4\sqrt{2}$$

(14) a)+b) <u>Circle</u>: $E = V'(r) = \ell^2/2mr^2 - k/r$

$$\frac{dV'}{dr} = 0 = -\ell^2/mr^3 + k/r^2 \qquad r_0 = \ell^2/mk$$

$$v^2 = v_\theta^2 = r^2 \dot{\theta}^2 = \ell^2/m^2 r^2 = \frac{\ell^2}{m^2} \left(\frac{mk}{\ell^2}\right)^2$$

$$v^2 = k^2/\ell^2 \qquad \therefore\ v_0 = \frac{k}{\ell} = \ell/mr_0$$

<u>Parabola</u> $$\frac{1}{r} = \frac{mk}{\ell^2}\left[1 + \cos\theta\right] \qquad \theta' = 0$$

perihelion: (r_{min}) $\quad \cos\theta = 1 \quad \frac{1}{r_{min}} = \frac{2mk}{\ell^2} \quad r_{min} = \frac{\ell^2}{2mk} = \frac{r_0}{2}$

Points of Intersection: $\frac{1}{r_0} = \frac{mk}{\ell^2} = \frac{1}{r_{par}} = \frac{mk}{\ell^2}(1+\cos\theta)$

$\cos\theta = 0 \qquad \theta = \pi/2,\ 3\pi/2$

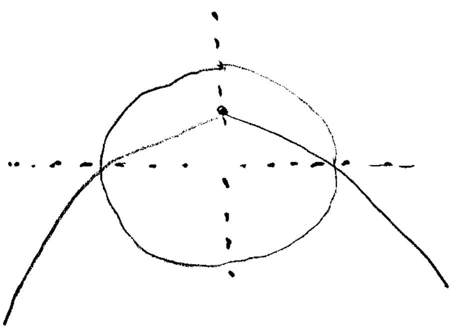
$e = 1 \Rightarrow E = 0$ since $e = \sqrt{1 + 2E\ell^2/mk^2} \qquad \ell \neq 0$

$E = 0 = \frac{1}{2}mv^2 - k/r$

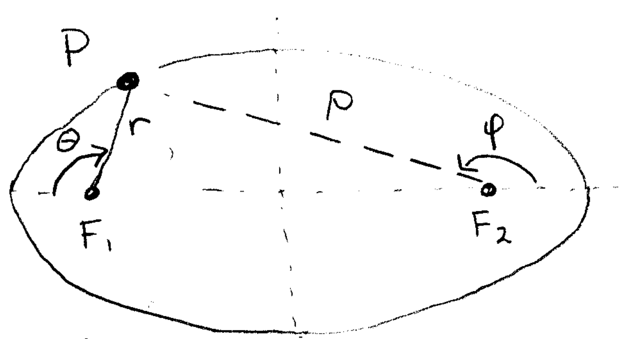
$v^2 = \frac{2k}{m}\frac{1}{r} = \frac{2k}{m}\frac{mk}{\ell^2}(1+\cos\theta)$

$v_p^2 = 2k^2/\ell^2\,(1+\cos\theta)$

$\cos\theta = 0 \Rightarrow v_p^2 = 2k^2/\ell^2 \qquad v_p = \sqrt{2}\,k/\ell = \sqrt{2}\,v_0$



(16)



$r + \rho = 2a$

$\dot{r} + \dot{\rho} = 0$

$\dot{\rho} = -\dot{r}$

$\dot{\theta} = \ell/mr^2 \qquad v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2$

$\therefore r^2\dot{\theta}^2 = \rho^2\dot{\varphi}^2$

$\dot{\varphi}^2 = \frac{r^2}{\rho^2}\dot{\theta}^2$

As drawn:

$\dot{\varphi} = -\frac{r}{\rho}\dot{\theta}$

$\dot{\varphi} = -\frac{r}{\rho}\frac{\ell}{mr^2} = -\frac{\ell}{mr\rho}$

$$\dot{\varphi} = -\frac{\ell}{mr}\,\frac{1}{2a-r} \qquad \frac{1}{r} = \frac{mk}{\ell^2}\left(1+\epsilon\cos\theta\right)$$

$$\dot{\varphi} = -\frac{\ell}{m}\left[\frac{mk}{\ell^2}\left(1+e\cos\theta\right)\right]\frac{1}{2a-\left[\frac{mk}{\ell^2}\left(1+e\cos\theta\right)\right]^{-1}}$$

$$\dot{\varphi} = -\frac{mk^2}{\ell^3}\left(1+e\cos\theta\right)^2\frac{1}{\frac{2amk}{\ell^2}\left(1+e\cos\theta\right)-1}$$

$$\left[\begin{array}{l} \text{Let } x = e\cos\theta \quad e\to 0 \quad x\to 0 \\ \dot{\varphi} = -\dfrac{mk^2}{\ell^3}\left(1+x\right)^2\dfrac{1}{\frac{2amk}{\ell^2}\left(1+x\right)-1} \end{array}\right]$$

For an ellipse $E = \frac{-k}{2a}$ $\quad e = \sqrt{1 + 2E\ell^2/mk^2}$

$$e = \sqrt{1-\ell^2/mka} \qquad e^2 = 1-\ell^2/mka$$

$$\ell^2/mka = 1-e^2$$

$$\dot{\varphi} = -\frac{mk^2}{\ell^3}\left(1+e\cos\theta\right)^2\frac{1}{\frac{2}{1-e^2}\left(1+e\cos\theta\right)-1}$$

$e^2 \to 0$

$$\dot{\varphi} \approx -\frac{mk^2}{\ell^3}\left(1+2e\cos\theta+\cdots\right)\frac{1}{2\left(1+e\cos\theta\right)-1}$$

$$= -\frac{mk^2}{\ell^3}\left(1+2e\cos\theta\right)\frac{1}{1+2e\cos\theta}$$

$$\dot{\varphi} \approx -\frac{mk^2}{\ell^3} \quad \text{uniform to 1ST order in } e$$

(19) a) Use conservation of energy (1ST integral)

$$E = \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) \ ; \ V(r) = -\frac{k}{r} e^{-r/a}$$

$k, a > 0$

$\dot{\theta} = \ell/mr^2$

$$V'(r) = V(r) + \ell^2/2mr^2$$

$$V'(r) = -\frac{k}{r} e^{-r/a} + \ell^2/2mr^2$$

First find extrema:

$$\frac{dV'}{dr} = 0 = \frac{k}{ra} e^{-r/a} + \frac{k}{r^2} e^{-r/a} - \ell^2/mr^3$$

$$k e^{-r/a} \left\{ \frac{1}{ra} + \frac{1}{r^2} \right\} = \ell^2/mr^3$$

$$k e^{-r/a} \left\{ r^2/a + r \right\} = \ell^2/m$$

$$e^{-r/a} \left\{ r^2/a + r \right\} = \ell^2/mk = r_0 \quad \text{(circle)}$$

Let $f(r) = e^{-r/a} \left\{ r^2/a + r \right\} = e^{-r/a} r \left[r/a + 1 \right]$

$r = 0, r = \infty$
$f(r) = 0$

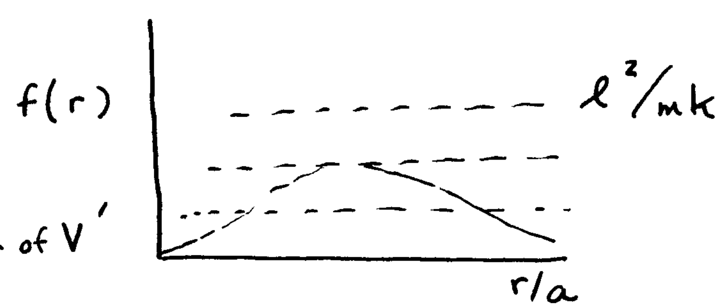
$$f'(r) = e^{-r/a} \left\{ -\frac{r}{a} \left[\frac{r}{a} + 1 \right] + \left[r/a + 1 \right] + r/a \right\}$$

$$f'(r) = 0 = \left\{ -(r/a)^2 + (r/a) + 1 \right\} = 0$$

$(r/a)^2 - (r/a) - 1 = 0$ maximum at $\frac{r_m}{a} = \frac{1 + \sqrt{5}}{2}$

$\therefore r_m = 1.618a$

& $f(r_m) = 0.840a$

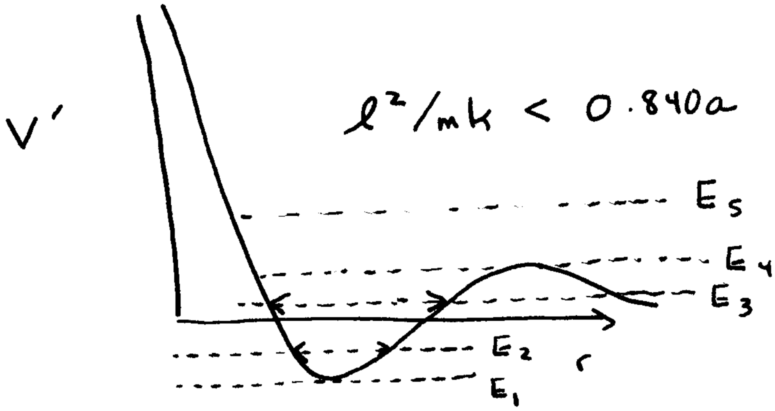


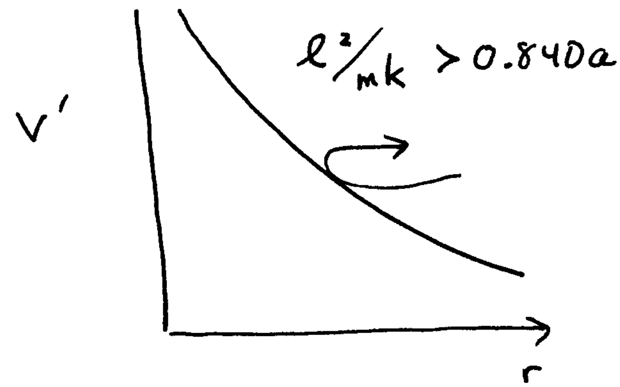
$\therefore$ If $\frac{\ell^2}{mk} > 0.840a$ no extremum of V'

If $\ell^2/mk = 0.840a$ one extremum

If $\ell^2/mk < 0.840a$ 2 extrema

From Mathematica results:





unbound for all E

E_5 : unbounded

E_4 : possibility of metastable circular orbit

E_3 : bounded or unbounded

E_2 : bounded

E_1 : stable circle

Plots of V' shown on next page

$$x = r/a$$

$$V'(x) = \frac{C}{x^2} - \frac{e^{-x}}{x}$$

$$C = \ell^2/2mak$$

(b) $u = u_0 + a\cos\beta\theta \qquad \beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\Big|_{r_0}$

$$V = -\frac{k}{r}e^{-r/a} \qquad f = -\frac{dV}{dr}$$

$$\therefore\ f = \frac{d}{dr}\left(\frac{k}{r}e^{-r/a}\right) = e^{-r/a}\left\{-\frac{k}{r^2} - \frac{k}{ar}\right\} = -ke^{-r/a}\left[r^{-2} + \frac{r^{-1}}{a}\right]$$

$$\frac{df}{dr} = -ke^{-r/a}\left\{\left(-\frac{1}{a}\right)\left[\frac{1}{r^2} + \frac{1}{ra}\right] + \left[-\frac{2}{r^3} - \frac{1}{ar^2}\right]\right\}$$

$$= ke^{-r/a}\left\{\frac{1}{ar^2} + \frac{1}{ra^2} + \frac{2}{r^3} + \frac{1}{ar^2}\right\}$$

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In[21]:= Plot[0.10 / (x*x) - E^(-x) / x, {x, 0.1, 5.0}]
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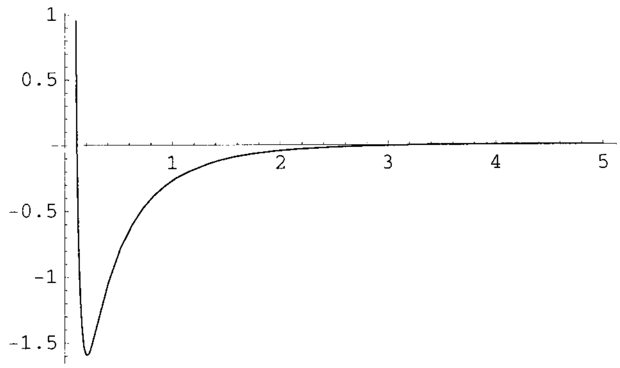


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Out[21]= - Graphics -
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In[23]:= Plot[0.35 / (x*x) - E^(-x) / x, {x, 0.1, 5.0}]
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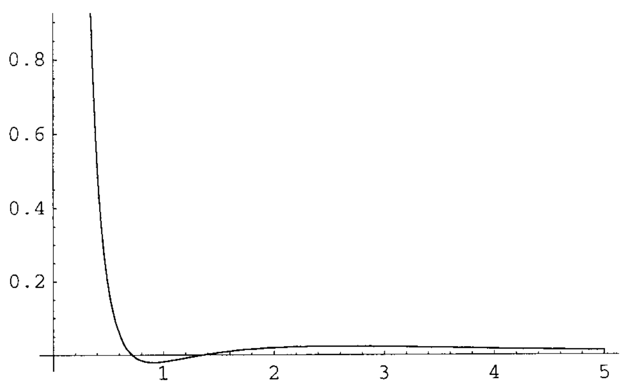


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Out[23]= - Graphics -
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In[24]:= Plot[0.50 / (x*x) - E^(-x) / x, {x, 0.1, 5.0}]
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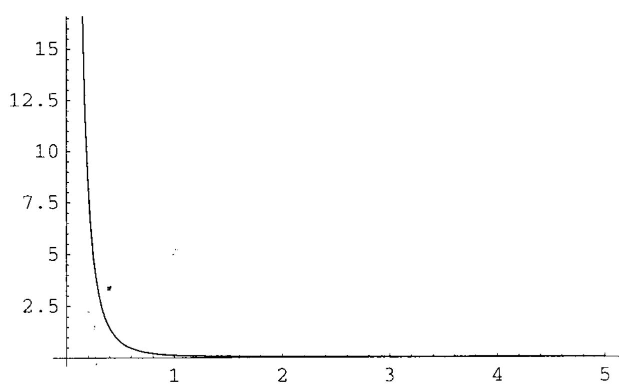


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Out[24]= - Graphics -
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$$\frac{df}{dr} = k e^{-r/a} \left\{ \frac{2}{r^3} + \frac{2}{ar^2} + \frac{1}{ra^2} \right\}$$

$$\beta^2 = 3 + \frac{r_0}{-k e^{-r_0/a} \left[\frac{1}{r_0^2} + \frac{1}{r_0 a} \right]} \cdot k e^{-r_0/a} \left[\frac{2}{r_0^3} + \frac{2}{a r_0^2} + \frac{1}{r_0 a^2} \right]$$

$$\beta^2 = 3 - \frac{\left[2/r_0^2 + 2/a r_0 + 1/a^2 \right]}{\frac{1}{r_0^2} + \frac{1}{r_0 a}}$$

$$\beta^2 = 3 - \frac{2 + 2\, r_0/a + (r_0/a)^2}{1 + r_0/a}$$

$r_0 << a \qquad (1 + r_0/a)^{-1} \approx 1 - r_0/a + \cdots$

$$\beta^2 \approx 3 - \left(2 + 2\, r_0/a + (r_0/a)^2\right)\left(1 - r_0/a + \left(\frac{r_0}{a}\right)^2 \cdots\right)$$

$$\beta^2 \approx 3 - 2 - 2\, r_0/a + 2 r_0/a - (r_0/a)^2 - 2(r_0/a)^2 + 2(r_0/a)^2$$

$$\beta^2 \approx 1 - (r_0/a)^2 \qquad \therefore \ \beta \approx 1 - \frac{1}{2}\left(\frac{r_0}{a}\right)^2$$

As θ advances by 2π $\beta\theta$ advances by

$$2\pi - \pi\,(r_0/a)^2$$

which means that the starting point has not yet been reached. The argument $\beta\theta$ reaches 2π when

$$\beta\theta = 2\pi \quad \text{or} \quad \theta = \frac{2\pi}{\beta} \approx 2\pi + 2\frac{\pi}{2}\,(r_0/a)^2$$

$\therefore$ The apsides <u>advance</u> by $\pi\,(r_0/a)^2$ per revolution