Physics 821

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Answers To Assignment # 2

Goldstein et al. Chapter 2

(D3) $$I = \int_1^2 \sqrt{(dx)^2+(dy)^2+(dz)^2} = \int_1^2 \underbrace{\sqrt{1+\dot{y}^2+\dot{z}^2}}_{f(\dot{y},\dot{z})}\, dx$$

$\delta I = 0 \Rightarrow$

$$\underbrace{\frac{\partial f}{\partial y}}_{0} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0 \quad + \quad \underbrace{\frac{\partial f}{\partial z}}_{0} - \frac{d}{dx}\frac{\partial f}{\partial \dot{z}} = 0$$

$$\frac{\partial f}{\partial \dot{y}} = \text{const} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2+\dot{z}^2}} = a \; ; \quad \frac{\partial f}{\partial \dot{z}} = \frac{\dot{z}}{\sqrt{1+\dot{y}^2+\dot{z}^2}} = b$$

$$\dot{y} = a(1+\dot{y}^2+\dot{z}^2)^{1/2} \qquad \dot{z} = b(1+\dot{y}^2+\dot{z}^2)^{1/2}$$

$$\dot{y}^2 = a^2(1+\dot{y}^2+\dot{z}^2) \qquad \dot{z}^2 = b^2(1+\dot{y}^2+\dot{z}^2)$$

$$\dot{y}^2(1-a^2) = a^2 + a^2\dot{z}^2 \qquad \dot{z}^2(1-b^2) = b^2 + b^2\dot{y}^2$$

$$\therefore \; \dot{y}^2(1-a^2) = a^2 + a^2\left[\frac{b^2+b^2\dot{y}^2}{1-b^2}\right]$$

$$\dot{y}^2(1-a^2)(1-b^2) = a^2(1-b^2) + a^2b^2 + a^2b^2\dot{y}^2$$

$$\dot{y}^2\left[1-a^2-b^2\right] = a^2(1-b^2) + a^2b^2 = a^2$$

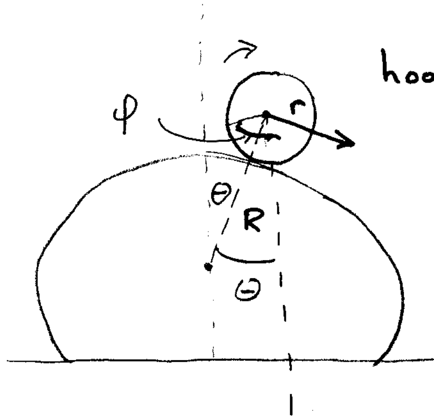
$$\dot{y}^2 = \frac{a^2}{1-a^2-b^2} \qquad \dot{y} = \frac{a}{\sqrt{1-a^2-b^2}} \equiv m \qquad y = mx + d$$

Similarly $$\dot{z}^2 = \frac{b^2}{1-a^2-b^2} \qquad \dot{z} = \frac{b}{\sqrt{1-a^2-b^2}} \equiv n \qquad z = nx + c$$

m, n, d, c determined by points 1 + 2.

Note: the equations of a straight line in 3-D space are $y = mx + d$ $\quad z = nx + c$

(14)



hoop of mass m

Constraints

1. $R\theta = r(\varphi - \theta)$

$f_1 = (R+r)\theta - r\varphi = 0$

$\dfrac{\partial f_1}{\partial \theta} = R + r \qquad \dfrac{\partial f_1}{\partial \varphi} = -r$

2. $R + r = \rho$

$f_2 = R + r - \rho = 0$

$\dfrac{\partial f_2}{\partial \rho} = -1$

Initial Conditions

$\varphi_0 = 0 \quad \dot\varphi_0 = 0$

$\theta_0 = 0 \quad \dot\theta_0 = 0$

Note: φ is departure from vertical on hoop

$$Q_k = -\sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k}$$

$$T = T_{cm} + T_{\substack{\text{about}\\ cm}} = \frac{m}{2}\rho^2\dot\theta^2 + \frac{1}{2}I\dot\varphi^2$$

$I = mr^2$

variables: $\theta, \varphi,$ "ρ" (leave out $\dot\rho$ term)

$$V = mg\rho\cos\theta \Rightarrow L = \frac{m}{2}\rho^2\dot\theta^2 + \frac{m}{2}r^2\dot\varphi^2 - mg\rho\cos\theta$$

$Q_\theta = -\lambda_1(R+r) \qquad Q_\varphi = \lambda_1 r \qquad Q_\rho = \lambda_2$ radial constraint

Note: constraint 1 is integrated form of $R\,d\theta = r(d\varphi - d\theta) \qquad df_1 = (R+r)\,d\theta - r\,d\varphi$

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ρ eq.: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot\rho} - \dfrac{\partial L}{\partial \rho} = \lambda_2 \qquad \dfrac{\partial L}{\partial \dot\rho} = 0 \qquad \dfrac{\partial L}{\partial \rho} = m\rho\dot\theta^2 - mg\cos\theta$

$$\therefore \boxed{-m\rho\dot\theta^2 + mg\cos\theta = \lambda_2} \qquad (1)$$

θ eq.: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot\theta} - \dfrac{\partial L}{\partial \theta} = -\lambda_1\rho \qquad \dfrac{\partial L}{\partial \dot\theta} = m\rho^2\dot\theta \qquad \dfrac{\partial L}{\partial \theta} = mg\rho\sin\theta$

$$\therefore \boxed{m\rho^2\ddot\theta - mg\rho\sin\theta = -\lambda_1\rho} \qquad (2)$$

φ eq.: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot\varphi} - \dfrac{\partial L}{\partial \varphi} = +\lambda_1 r \qquad \dfrac{\partial L}{\partial \dot\varphi} = mr^2\dot\varphi \qquad \dfrac{\partial L}{\partial \varphi} = 0$ "cyclic"

$$mr^2\ddot\varphi = \lambda_1 r \quad \text{or} \quad \boxed{mr\ddot\varphi = \lambda_1} \qquad (3)$$

Idea is to find $Q_\rho = \lambda_2$ and set it to zero.

From constraint (1): $\rho\theta = r\varphi \Rightarrow \ddot\theta = \dfrac{r}{\rho}\ddot\varphi = \dfrac{1}{\rho}\dfrac{\lambda_1}{m}$

From eq. (2): $\dfrac{m\rho^2}{\rho}\dfrac{\lambda_1}{m} - mg\rho\sin\theta = -\lambda_1\rho$

$$2\lambda_1\rho = mg\rho\sin\theta \; ; \quad \lambda_1 = \frac{mg}{2}\sin\theta$$

$$\therefore \ddot\theta = \frac{1}{\rho}\frac{\lambda_1}{m} = g\sin\theta/2\rho$$

$$\dot\theta\ddot\theta = \frac{g}{2\rho}\sin\theta\,\dot\theta \qquad \frac{1}{2}\frac{d}{dt}(\dot\theta^2) = -\frac{g}{2\rho}\frac{d}{dt}(\cos\theta)$$

$$\dot\theta^2 = -\frac{g}{\rho}\cos\theta + K \qquad K = g/\rho \quad \text{from } \dot\theta_0 = 0$$

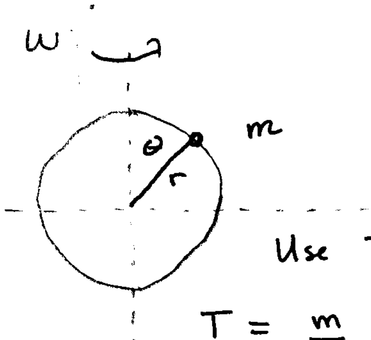
Now: $\lambda_2 = -m\rho\dot\theta^2 + mg\cos\theta \qquad (1)$

$$\lambda_2 = -m\rho\left[-\frac{g}{\rho}\cos\theta + \frac{g}{\rho}\right] + mg\cos\theta$$

$$\lambda_2 = 2mg\cos\theta - mg = 0$$

$$\cos\theta = 1/2 \qquad \theta = 60^\circ$$

(18)



Use standard generalized coordinate method.

$\omega = \dot\varphi \qquad r = a$

$\varphi = \omega t$

only coordinate needed is θ

Use T from spherical polar coordinates

$r = a$

$$T = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2\right)$$

$$T = \frac{m}{2}\left(a^2\dot\theta^2 + a^2\omega^2\sin^2\theta\right) \qquad V = mga\cos\theta$$

18 cont.
$$L = \frac{m}{2}\left[a^2 \dot{\theta}^2 + a^2 \omega^2 \sin^2\theta \right] - mga\cos\theta$$

(i) $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot\theta} = ma^2\dot\theta$

$$\frac{\partial L}{\partial \theta} = \frac{m}{2} a^2 \omega^2 \, 2\sin\theta\cos\theta + mga\sin\theta$$
$$= ma^2\omega^2 \sin\theta\cos\theta + mga\sin\theta$$

$$\therefore \; ma^2\ddot\theta - ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta = 0$$
$$\boxed{\ddot\theta - \omega^2\sin\theta\cos\theta - g/a\,\sin\theta = 0}$$

(ii) Constants of The Motion

L is not an explicit function of t

$$\therefore \; \frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0 \qquad h \text{ conserved}$$

What is h? $\quad h = \sum_j \dot q_j \frac{\partial L}{\partial \dot q_j} - L$

$$h = \dot\theta\frac{\partial L}{\partial\dot\theta} - L = ma^2\dot\theta^2 - \frac{m}{2}a^2\dot\theta^2 - \frac{m}{2}a^2\omega^2\sin^2\theta + \underbrace{mga\cos\theta}_{V}$$

$$h = \frac{m}{2}a^2\dot\theta^2 - \frac{m}{2}a^2\omega^2\sin^2\theta + V$$

$$E = T + V = \frac{m}{2}a^2\dot\theta^2 + \frac{m}{2}a^2\omega^2\sin^2\theta + V \neq h$$

Reason: one transformation ($\varphi = \omega t$) is time-dependent or T contains T_0 term $\frac{m}{2}a^2\omega^2\sin^2\theta$

In fact, $E = h + ma^2\omega^2\sin^2\theta$ + is not a constant of the motion.

What about p_θ?
Lagrange's equation can be written as

$$\frac{d}{dt}(p_\theta) = \frac{\partial L}{\partial\theta} \neq 0 \qquad p_\theta \text{ not conserved}$$

(iii) Stationary Points Θ_s

$$\Theta = \Theta_s \qquad \dot{\Theta} = \ddot{\Theta} = 0$$

$$\ddot{\Theta} = 0 \Rightarrow \quad -\omega^2 \sin\Theta_s \cos\Theta_s - g/a \sin\Theta_s = 0$$

$$-\omega^2 \sin\Theta_s \cos\Theta_s = g/a \sin\Theta_s$$

$\sin\Theta_s \neq 0 \qquad -\omega^2 \cos\Theta_s = g/a \qquad \cos\Theta_s = \dfrac{-g}{a\omega^2} < 0$

$\therefore \ \Theta_s > \dfrac{\pi}{2} \qquad \boxed{\Theta_s = \cos^{-1}\left(-g/a\omega^2\right)}$

$\cos\Theta_s \geq -1 \qquad -\dfrac{g}{a\omega^2} \geq -1 \qquad \dfrac{g}{a\omega^2} \leq 1$

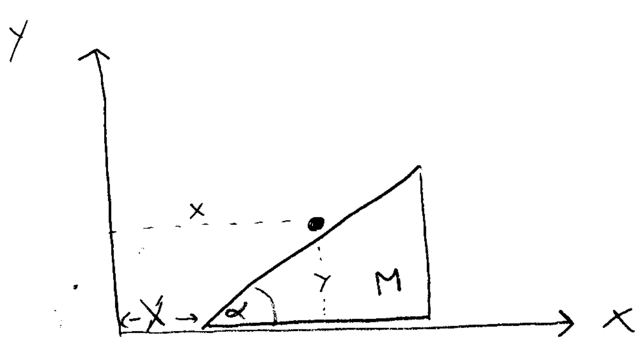
$$\therefore \ \omega^2 \geq g/a \qquad \omega \geq \sqrt{g/a} = \omega_0$$

Other solution: $\sin\Theta_s = 0 \qquad \Theta_s = \pi$ ($\Theta_s = 0$ not stable)

Note: equation of motion can be integrated by multiplication by $\dot{\Theta}$:

$$\frac{d}{dt}\Big[\underbrace{\frac{1}{2}\dot{\Theta}^2 - \frac{\omega^2}{2}\sin^2\Theta + \frac{g}{a}\cos\Theta}_{h/ma^2}\Big] = 0$$

(20)



Constraint

$$x = X + y\cot\alpha$$

$$f = X + y\cot\alpha - x = 0$$

(i) Equations of Motion

x, y, X coordinates

$$Q_k = -\lambda \frac{\partial f}{\partial q_k}$$

$$Q_X = -\lambda$$

$$Q_x = +\lambda$$

$$Q_y = -\lambda\cot\alpha$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad V = mgy$$

$$L = \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

x equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x = +\lambda;\quad m\ddot{x} = \lambda \qquad (1)$

y equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = Q_y = -\lambda\cot\alpha;\quad m\ddot{y} + mg = -\lambda\cot\alpha \qquad (2)$

X equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = Q_X = -\lambda;\quad M\ddot{X} = -\lambda \qquad (3)$

constraint : $X + y\cot\alpha - x = 0 \qquad (4)$

unknowns : X, x, y, λ

(ii) <u>Solution For Forces of Constraint</u>

(4) $\Rightarrow$ $\ddot{y}\cot\alpha = \ddot{x} - \ddot{X} = \frac{\lambda}{m} + \frac{\lambda}{M} = \frac{\lambda}{\mu}$ $\qquad \mu^{-1} = m^{-1} + M^{-1}$ reduced mass

$\therefore\ \lambda = \mu\ddot{y}\cot\alpha$

(2) $\Rightarrow$ $m\ddot{y} + mg = -\mu\ddot{y}\cot^2\alpha$

$$\ddot{y}(m + \mu\cot^2\alpha) = -mg$$

$$\ddot{y} = \frac{-g}{1 + \mu/m\cot^2\alpha} = \frac{-g}{1 + \frac{M}{m+M}\cot^2\alpha}$$

$$\therefore\ \lambda = \mu\ddot{y}\cot\alpha = \frac{-\mu g\cot\alpha}{1 + \mu/m\cot^2\alpha}$$

$$Q_x = \lambda = \frac{-\mu g\cot\alpha}{1 + \frac{\mu}{m}\cot^2\alpha} = \frac{-\mu g\tan\alpha}{\tan^2\alpha + \mu/m}$$

$$Q_y = -\lambda\cot\alpha = \frac{\mu g\cot^2\alpha}{1 + \mu/m\cot^2\alpha} = \frac{\mu g}{\tan^2\alpha + \mu/m}$$

$$Q_X = -\lambda = \frac{\mu g\cot\alpha}{1 + \mu/m\cot^2\alpha} = \frac{\mu g\tan\alpha}{\tan^2\alpha + \mu/m}$$

(iii) Work Done In Time t By Forces of Constraint

$$dW = Q_x\,dx + Q_y\,dy + Q_X\,dX$$
$$dW = Q_x\,\dot{x}\,dt + Q_y\,\dot{y}\,dt + Q_X\,\dot{X}\,dt$$

Integrating equations of motion assuming no velocity at $t=0$

$$\dot{x} = \frac{\lambda}{m}t \qquad \dot{y} = \frac{\lambda}{\mu\cot\alpha}t \qquad \dot{X} = -\frac{\lambda}{M}t$$

$$Q_x = \lambda \qquad Q_y = -\lambda\cot\alpha \qquad Q_X = -\lambda$$

$$dW = \Big[\underbrace{\frac{\lambda^2}{m} - \frac{\lambda^2}{\mu} + \frac{\lambda^2}{M}}_{0}\Big]\,t\,dt$$

$$\therefore\quad W = 0$$

Individual Contributions :

$$W_x = \frac{\lambda^2}{2m}t^2 \qquad W_y = -\frac{\lambda^2}{2\mu}t^2 \qquad W_X = \frac{\lambda^2}{2M}t^2$$

$$\lambda = -\frac{\mu g\tan\alpha}{\tan^2\alpha + \mu/m}$$

(iv) Constants of Motion

Redo problem without multipliers. Put constraints into L.

Let's get rid of $X = x - y\cot\alpha$

a) $$L = \frac{M}{2}(\dot{x} - \dot{y}\cot\alpha)^2 + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

x cyclic $\quad \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad p_x$ conserved

but what is p_x ?

$$p_x = \frac{\partial L}{\partial \dot{x}} = M\dot{x} - M\dot{y}\cot\alpha + m\dot{x}$$
$$= \dot{x}(M+m) - M\cot\alpha\,[\,\dot{x} - \dot{X}\,]/\cot\alpha$$

$$P_x = \dot{x}(M+m) - M(\dot{x} - \dot{X}) = m\dot{x} + M\dot{X}$$

P_x is total momentum in horizontal direction (no applied forces)

(b) $\dfrac{dh}{dt} = -\dfrac{\partial L}{\partial t} = 0 \quad \therefore \quad h$ is conserved

$$h = L_2 - L_0 \qquad L_2 = T_2 = T \qquad L_0 = -V_0 = -V$$

$h = T + V = E$ conserved

(v) Fixed Wedge

$$\dot{X} = 0 \Rightarrow \ddot{X} = 0 \qquad \ddot{X} = -\frac{\lambda}{M} \Rightarrow M = \infty$$

$$\frac{1}{\mu} \to \frac{1}{m} \qquad \mu = m$$

$$\lambda = m\ddot{y}\cot\alpha \qquad \lambda = -\frac{mg\cot\alpha}{1+\cot^2\alpha} = -mg\sin\alpha\cos\alpha$$

$$\ddot{y} = -\frac{g}{1+\cot^2\alpha} \qquad \lambda = -\frac{mg}{2}\sin 2\alpha$$

$$Q_x = \lambda = -\frac{mg}{2}\sin 2\alpha \qquad Q_y = -\lambda\cot\alpha = mg\cos^2\alpha$$

$$dW = Q_x\dot{x}\,dt + Q_y\dot{y}\,dt$$

$$\dot{x} = \frac{\lambda}{m}t \qquad \dot{y} = \frac{\lambda t}{m\cot\alpha}$$

$$dW = \left[\frac{\lambda^2}{m} - \frac{\lambda^2}{m}\right]t\,dt = 0 \qquad W = 0$$

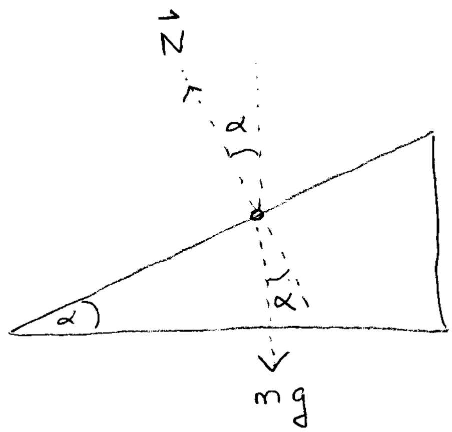
Constants of Motion

h still conserved and equal to energy

$P_x = m\dot{x}$ is NOT conserved

$L = \frac{m}{2}(\dot{x}^2 + \dot{x}^2\tan^2\alpha) - mgx\tan\alpha \qquad x$ not cyclic

Classical Analysis



$N = mg\cos\alpha$

$N_x = -N\sin\alpha = -mg\sin\alpha\cos\alpha$
(Q_x)

$N_y = N\cos\alpha = mg\cos^2\alpha \quad (Q_y)$

Extra Problem

$$L = \frac{1}{2}mv^2 - \underbrace{q\varphi}_{0} + q\vec{A}\cdot\vec{v} \qquad (1.63)$$

$$A_r = A_\theta = 0 \qquad A_z = -\frac{\mu_0 I}{2\pi}\ln r$$

$$\vec{A}\cdot\vec{v} = A_\theta r\dot{\theta} + A_r\dot{r} + A_z\dot{z} = A_z\dot{z}$$
$$= -\left(\frac{\mu_0 I}{2\pi}\ln r\right)\dot{z}$$

$$L = \frac{m}{2}\left(\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2\right) - q\left(\frac{\mu_0 I}{2\pi}\ln r\right)\dot{z}$$

θ, z cyclic coordinates $\quad \dfrac{\partial L}{\partial \theta} = \dfrac{\partial L}{\partial z} = 0$

P_θ, P_z conserved

$$\frac{\partial L}{\partial \dot{\theta}} = P_\theta = mr^2\dot{\theta} = L_z \quad \text{conserved}$$

$$\frac{\partial L}{\partial \dot{z}} = P_z = m\dot{z} - q\frac{\mu_0 I}{2\pi}\ln r \quad \text{conserved}$$