Physics 780.04 Winter 2009

Answers To H.W. Set 1

1. $\tau = A_{ul}^{-1} = 1 \times 10^{-6}$ s. $\lambda_{ul} = 2500\ \text{Å} = 2.5 \times 10^{-5}$ cm

$$\nu_{ul} = c/\lambda_{ul} = \frac{2.9979 \times 10^{10}\ \text{cm s}^{-1}}{2.500 \times 10^{-5}\ \text{cm}} = 1.199 \times 10^{15}\ \text{Hz}\ (\text{s}^{-1})$$

(a) $$A_{ul} = \frac{32\pi^3 \nu_{ul}^3}{3\hbar c^3} |(\mu)_{ul}|^2 \ ; \quad |(\mu)_{ul}|^2 = \frac{3\hbar c^3 A_{ul}}{32\pi^3 \nu_{ul}^3}$$

$$|(\mu)_{ul}|^2 = 4.984 \times 10^{-38}\ \text{erg cm}^3 \quad (\text{esu}^2\,\text{cm}^2)$$

$$|(\mu)_{ul}| = 2.2325 \times 10^{-19}\ \text{esu-cm} = 0.22325\ \text{Debye}$$

Note: in cgs units $F = e^2/r^2$ dyne cm^2 = esu^2

(b) $$\Delta\nu = \frac{1}{2\pi}\Gamma = \frac{1}{2\pi} A_{ul} = 1.592 \times 10^5\ \text{Hz} = 0.1592\ \text{MHz}$$

$$f_{ul} = \frac{8\pi^2 m_e \nu_{ul}}{3he^2} |(\mu)_{ul}|^2 = 0.00094$$

$\uparrow$ use esu

(c) $k_{coll} = 1 \times 10^{-10}\ \text{cm}^3\ \text{s}^{-1}$ $\quad \Delta\nu = \frac{1}{2\pi} k_{coll}\, n_{gas} = \gamma P(\text{torr})$

$P = nkT$ $\quad$ 1 atm $= 1.01325 \times 10^6$ dynes/cm^2

$$P(\text{dynes/cm}^2) = \frac{1}{760}\ \text{atm} \times 1.01325 \times 10^6\ \frac{\text{dyne/cm}^2}{\text{atm}} = 1.333 \times 10^3\ \frac{\text{dyne}}{\text{cm}^2}$$

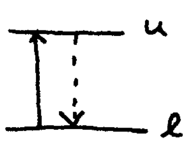
$$n_{gas} = P/kT = 3.22 \times 10^{16}\ \text{cm}^{-3}$$

$$\Delta\nu = 5.12 \times 10^5\ \text{Hz}\ (\text{s}^{-1}) = 0.512\ \text{MHz}$$

$P = 1$ torr $\Rightarrow$ $\gamma = 0.512$ MHz/Torr

(d) $$\sigma = \int \sigma_\nu d\nu = \frac{8\pi^3}{3hc} \nu_{ul} |(\mu)_{ul}|^2 = 2.487 \times 10^{-5}\ \text{cm}^2\ \text{s}^{-1}$$

$$K = \int K_\nu d\nu = \sigma n_l = 2.487 \times 10^9\ \text{cm}^{-1}\ \text{s}^{-1}$$

2. 

$$\int K_\nu d\nu = \frac{h\nu_{ul}}{c}\left[B_{u\leftarrow l}\, n_l - B_{u\rightarrow l}\, n_u\right]$$

$$g_l B_{u\leftarrow l} = g_u B_{u\rightarrow l}$$

$$\int K_\nu d\nu = \frac{h\nu_{ul}}{c}\left[B_{u\leftarrow l}\, n_l - \frac{g_l}{g_u} B_{u\leftarrow l}\, n_l \frac{g_u}{g_l} e^{-h\nu_{ul}/kT}\right]$$

(2) cont. $\int \kappa_\nu d\nu = \frac{h\nu_{u\ell}}{c} B_{u\leftarrow\ell} \left[\underbrace{1 - e^{-h\nu_{u\ell}/kT}}_{1-(1-h\nu_{u\ell}/kT) \approx h\nu_{u\ell}/kT} \right] n_\ell$

$$\int \kappa_\nu d\nu = \frac{(h\nu_{u\ell})^2}{ckT} B_{u\leftarrow\ell}\, n_\ell = \frac{h^2\nu_{u\ell}^2}{ckT}\,\frac{2\pi}{3\hbar^2}\,|(\mu)_{u\leftarrow\ell}|^2 n_\ell$$

$$K = \int \kappa_\nu d\nu = \frac{8\pi^3 \nu_{u\ell}^2}{3ckT}\, n_\ell\, |(\mu)_{u\leftarrow\ell}|^2$$

(3) $P_G(\nu) = e^{-mv^2/2kT} = e^{-m(\nu-\nu_{u\ell})^2 c^2/2\nu_{u\ell}^2 kT}$

$\frac{\nu-\nu_{u\ell}}{\nu_{u\ell}} = \frac{v}{c}$ HWHM $\frac{\Delta\nu}{2}$ defined by $P(\nu = \nu_{u\ell} \pm \frac{\Delta\nu}{2}) = \frac{1}{2}$

$$\frac{1}{2} = \exp\left\{ -m\left[\pm\frac{\Delta\nu}{2}\right]^2 c^2/2\nu_{u\ell}^2 kT \right\}$$

$$-\ln 2 = -m\left[\frac{\Delta\nu}{2}\right]^2 c^2/2\nu_{u\ell}^2 kT$$

$$\left(\frac{\Delta\nu}{2}\right)^2 = (\ln 2)\, 2\nu_{u\ell}^2 kT/mc^2$$

$$\frac{\Delta\nu}{2} = \frac{\nu_{u\ell}}{c}\sqrt{\frac{2kT\ln 2}{m}}$$

$m = m_c = 12/N_A$ $T = 300\,\text{K}$ $\frac{\Delta\nu}{\nu_{u\ell}} = 3.6 \times 10^{-6}$

$\Delta\nu = (3.6\times10^{-6})(1.199\times10^{15}\,\text{s}^{-1}) = 4.3\times10^{9}\,\text{s}^{-1} = 4.3\,\text{GHz}$

$\therefore$ much larger than collisional broadening here

(4) $\kappa_\nu = \kappa_{max} P(\nu)$ $K = \int \kappa_\nu d\nu$ $\kappa_\nu = K\, g(\nu-\nu_{u\ell})$

$\int \kappa_\nu d\nu = \kappa_{max} \int P(\nu)\, d\nu$

a) homogeneous case $P_L(\nu) = (\Delta\nu/2)^2 \Big/ (\nu-\nu_{u\ell})^2 + \left(\frac{\Delta\nu}{2}\right)^2$

$$\int P_L(\nu) d\nu = \int_{-\infty}^{\infty} P_L(\nu-\nu_{u\ell})\, d(\nu-\nu_{u\ell})$$

(4) cont.

$$\int P_L(\nu)d\nu = \left(\frac{\Delta\nu}{2}\right)^2 \underbrace{\int_{-\infty}^{\infty} \frac{d(\nu-\nu_{ul})}{(\nu-\nu_{ul})^2 + (\Delta\nu/2)^2}}_{\pi(\Delta\nu/2)^{-1}}$$

$$\therefore \int P_L(\nu)d\nu = \frac{\Delta\nu}{2}\pi$$

$$\int K_\nu d\nu = K_{max}\frac{\Delta\nu}{2}\pi \Rightarrow K_{max}\,(\text{homo}) = \frac{2}{\Delta\nu_L \pi}\int K_\nu d\nu$$

b) heterogeneous case

$$P_M(\nu) = e^{-\ln 2\,(\nu-\nu_{ul})^2/(\Delta\nu/2)^2}$$

Note: $\int_{-\infty}^{\infty} e^{-Ax^2}dx = \sqrt{\pi/A}$

$$\therefore \int_{-\infty}^{\infty} P_M(\nu-\nu_{ul})\,d(\nu-\nu_{ul}) = \int_{-\infty}^{\infty} e^{-\{\ln 2/(\Delta\nu/2)^2\}(\nu-\nu_{ul})^2}\,d(\nu-\nu_{ul})$$

$$= \pi^{1/2}\sqrt{\frac{(\Delta\nu/2)^2}{\ln 2}} = \frac{\Delta\nu}{2}\sqrt{\frac{\pi}{\ln 2}}$$

$$\therefore \int K_\nu d\nu = K_{max}\int P_G(\nu)d\nu = K_{max}\frac{\Delta\nu}{2}\sqrt{\frac{\pi}{\ln 2}}$$

$$\therefore K_{max}\,(\text{hetero}) = \frac{2}{\Delta\nu_G}\sqrt{\frac{\ln 2}{\pi}}\int K_\nu d\nu$$

The Gaussian K_{max} is the relevant one for problem 1 since the Doppler line width is much the larger.

$$\therefore K_{max} = \frac{2}{4.3\times10^9\ s^{-1}}\sqrt{\frac{\ln 2}{\pi}}\ 2.487\times10^9\ cm^{-1}s^{-1}$$

$$= 0.543\ cm^{-1}$$

$I(\nu_{ul}) = I_0(\nu)\,e^{-K_\nu^{max}\ell}$ at resonance

$I(\nu_{ul})/I_0(\nu) = \exp[-0.543] = 0.581$

Absorbed % = 0.419 or 41.9%

(5) Ch 1 Problem 7

a) $$p(\nu) = \frac{8\pi h \nu^3}{c^3}\left(e^{h\nu/kT} - 1\right)^{-1}$$

$$\frac{dp(\nu)}{d\nu} = 0 = \frac{8\pi h}{c^3}\left[3\nu^2\left(e^{h\nu/kT}-1\right)^{-1} + \nu^3(-1)\left(e^{h\nu/kT}-1\right)^{-2}\frac{h}{kT}e^{h\nu/kT}\right]$$

$$3\nu^2\left(e^{h\nu/kT}-1\right) = \nu^3 h/kT\; e^{h\nu/kT}$$

$$3\left(e^{h\nu/kT}-1\right) = h\nu/kT\; e^{h\nu/kT} \qquad \text{Let } x = h\nu/kT$$

$$3(e^x - 1) = x e^x$$ Solved by Mathematica $x = 2.8214$

$\therefore p_\nu$ is maximum at $\nu_{max} = 2.8214\, kT/h$

b) $p_\nu d\nu = -p_\lambda d\lambda \qquad c = \lambda\nu \qquad \nu = c/\lambda \qquad d\nu = -\frac{c}{\lambda^2}d\lambda$

$$\therefore p_\lambda = -p_\nu \frac{d\nu}{d\lambda} = p_\nu \frac{c}{\lambda^2} = \frac{8\pi h}{c^3}\left(\frac{c}{\lambda}\right)^3 \frac{1}{e^{hc/\lambda kT}-1}\,\frac{c}{\lambda^2}$$

$$\therefore p_\lambda = \frac{8\pi h c}{\lambda^5}\,\frac{1}{e^{hc/\lambda kT}-1}$$

c) Proceeding as in (a):

$$\frac{dp_\lambda}{d\lambda} = 0 = 8\pi hc\left\{\frac{-5\lambda^{-6}}{e^{hc/\lambda kT}-1} + \lambda^{-5}(-1)\left(e^{hc/\lambda kT}-1\right)^{-2} \times e^{hc/\lambda kT}\left(-\frac{hc}{\lambda^2 kT}\right)\right\}$$

$$5\left[e^{hc/\lambda kT} - 1\right] = \frac{hc}{\lambda kT}\, e^{hc/\lambda kT} \qquad x = hc/\lambda kT$$

$5(e^x - 1) = x e^x$ Mathematica $\Rightarrow x = 4.9651$

$$\therefore \frac{hc}{\lambda_{max} kT} = 4.9651 \Rightarrow \lambda_{max} T = \frac{hc}{k\, 4.9651}$$

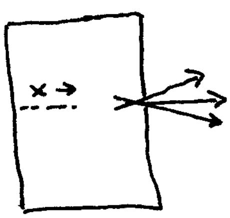
mks units : $hc/k = 1.439 \times 10^{-2}$ m-K

$$\therefore \quad \lambda_{max} T = 2.898 \times 10^{-3} \text{ m-K}$$

(d)

T (K)	λ_{max}	
3	9.659×10^{-4} m or 0.09659 cm	interstellar space (radio)
293	9.890×10^{-6} m or 9.890 μm	room temp (IR)
2273	1.275×10^{-6} m or 1.275 μm or 1275 nm	flame (near IR)
6000K	4.830×10^{-7} m or 483.0 nm	solar photosphere (visible; green)

(6) Problem 8 Ch. 1

a) 

$$I(\nu) = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{c\,p(\nu)}{4\pi} \underbrace{\cos\theta}_{\text{projection onto x-axis}} \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega}$$

$$I(\nu) = \frac{2\pi}{4\pi} c\,p(\nu) \int_{\theta=0}^{\pi/2} \underbrace{\cos\theta \sin\theta \, d\theta}_{\frac{1}{2}\sin^2\theta \Big|_0^{\pi/2} = \frac{1}{2}} = \frac{1}{4} c\,p(\nu)$$

$$I = \int_0^\infty I(\nu) d\nu = \frac{c}{4} \int_0^\infty \frac{8\pi h}{c^3} \nu^3 \frac{1}{e^{h\nu/kT} - 1} d\nu$$

$$x = h\nu/kT \qquad dx = \frac{h}{kT} d\nu$$

$$\therefore \quad I = \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^3 \int_{x=0}^\infty x^3 \frac{1}{e^x - 1} \frac{kT}{h} dx$$

$$I = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_{x=0}^{\infty} \frac{x^3}{e^x - 1}\,dx = \sigma T^4$$

(b) $$\sigma = \frac{2\pi h}{c^2}\frac{k^4}{h^4}\underbrace{\int_0^{\infty}\frac{x^3}{e^x-1}\,dx}_{\pi^4/15} = \frac{2\pi^5 k^4}{15 c^2 h^3}$$

mks units: $$\frac{J^4 K^{-4}}{m^2 s^{-2} J^3 s^3} = \frac{J K^{-4}}{m^2 \cdot s} = W\,m^{-2}\,K^{-4}$$

$$\sigma = \frac{2\pi^5 (1.38065 \times 10^{-23}\,J/K)^4}{15\,(2.99792 \times 10^8\,m/s)^2\,(6.6261 \times 10^{-34}\,J\,s)^3}$$

$$\sigma = 5.670 \times 10^{-8}\ W\,m^{-2}\,K^{-4}$$

(7) Ch 1 Problem 10

E_1 ———

E_0 ———

$n_1/n_0 = e^{-\Delta E_{1,0}/kT}$ $\qquad \Delta E = h\nu = \frac{hc}{\lambda} = hc\tilde{\nu}$

$\lambda = 6000\,\text{Å} = 6 \times 10^{-5}\,cm \Rightarrow \Delta E = 3.311 \times 10^{-12}\,erg$
(red)

$\tilde{\nu} = 1000\,cm^{-1} \Rightarrow \Delta E = 1.986 \times 10^{-13}\,erg$
(IR)

$\nu = 100\,GHz = 1.00 \times 10^{11}\,s^{-1} \Rightarrow \Delta E = 6.626 \times 10^{-16}\,erg$
(μwave)

$\nu = 1\,GHz = 1.00 \times 10^{9}\,s^{-1} \Rightarrow \Delta E = 6.626 \times 10^{-18}\,erg$
(radio)

ΔE (erg)	n_1/n_0 (293 K)	n_1/n_0 (6000 K)
3.311×10^{-12} (red)	2.84×10^{-36}	0.0184
1.986×10^{-13} (IR)	0.00738	0.787
6.626×10^{-16} (μwave)	0.984	0.9992
6.626×10^{-18} (radio)	0.9998	0.999992