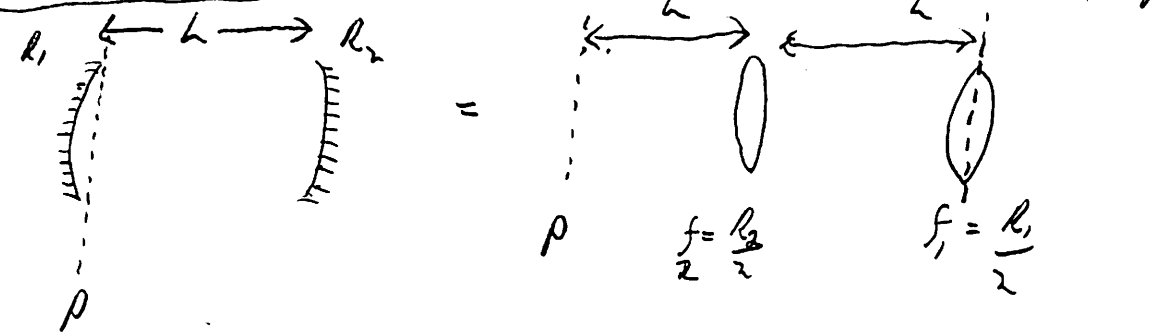
Two mirror Resonator



Make the signs make sense.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/R_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2/R_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & L \\ -\frac{2}{R_2} & -\frac{2L}{R_2}+1 \end{bmatrix}$$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1-\frac{2L}{R_2} & L-\frac{2L^2}{R_2}+L \\ -\frac{2}{R_2} & -\frac{2L}{R_2}+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{2L}{R_2} & \sim \\ \sim & -\frac{2L}{R_1}+\frac{4L^2}{R_2R_1}-\frac{2L}{R_1}-\frac{2L}{R_2}+1 \end{bmatrix}$$

$$A+D = 2-\frac{4L}{R_2}-\frac{4L}{R_1}+\frac{4L^2}{R_1R_2}$$

$$\frac{A+D}{2} = 1-\frac{2L}{R_1}-\frac{2L}{R_2}+\frac{2L^2}{R_1R_2}$$

$$\frac{A+D}{2} = 2\left[1-\left(\frac{L}{R_1}\right)\right]\left[1-\left(\frac{L}{R_2}\right)\right]-1$$

$$= 2g_1g_2-1$$

$$\left(\frac{A+D}{2}\right)^2 \le 1$$

$$-1 \le \frac{A+D}{2} \le 1$$

$$2g_1g_2-1 \le 1 \Rightarrow g_1g_2 \le 1$$

$$2g_1g_2-1 \ge -1 \Rightarrow g_1g_2 \ge 0$$

$$g_1 \equiv 1-\frac{L}{R_1}$$

$$g_2 \equiv 1-\frac{L}{R_2}$$

$$\left(\frac{A+D}{2}\right)^2 \le 1 \Rightarrow 0 \le g_1g_2 \le 1$$

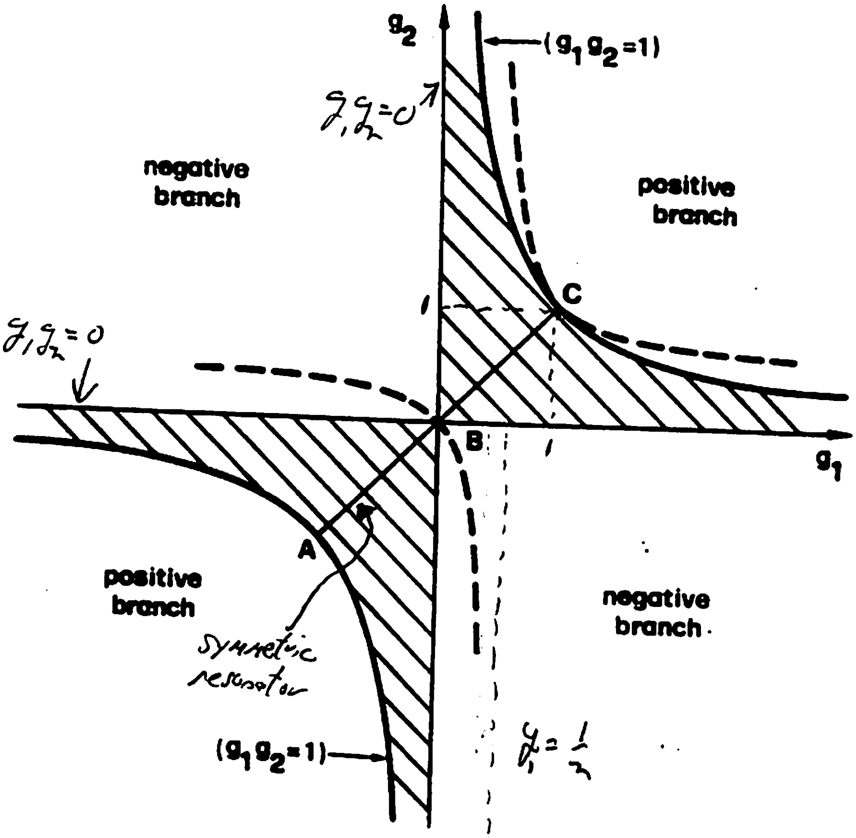


FIG. 5.7. A g_1, g_2 stability diagram for a general spherical resonator. The stable region corresponds to shaded parts in the figure; dashed curves correspond to the possible confocal resonators.

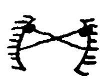
Two meanings of stable:

① "stable mode" – a consistent mode exists,

② "stable against perturbations" = robust

HW problems 4+5.

plane-parallel  $R_1 = R_2 = \infty$ ⇒ $g_1 = 1$, $g_2 = 1$ "C"

confocal  $R_1 = R_2 = L$ $g_1 = 0$, $g_2 = 0$ "B"

concentric  $R_1 = R_2 = \frac{L}{2}$ $g_1 = -1$, $g_2 = -1$ "A"

Marginally stable

More generally ⇒ $g_1 g_2 = 0$ & $g_1 g_2 = 1$

Stable modes exist but, in the ray picture, only rays that are normal to the mirror surface.

Confocal case ⇒ $L = \frac{R_1}{2} + \frac{R_2}{2}$

$$g_1 = 1 - \frac{1}{2} - \frac{R_2}{2R_1} = \frac{1}{2}\left(1 - \frac{R_2}{R_1}\right)$$

$$g_2 = 1 - \frac{1}{2} - \frac{R_1}{2R_2} = \frac{1}{2}\left(1 - \frac{R_1}{R_2}\right)$$

$$1 - 2g_1 = \frac{R_2}{R_1}$$

$$g_2 = \frac{1}{2}\left(1 - \frac{1}{1-2g_1}\right) = \frac{1}{2}\left(\frac{-2g_1}{1-2g_1}\right)$$

$$g_2 = \frac{g_1}{2g_1 - 1}$$

$$\frac{1}{g_2} = 2 - \frac{1}{g_1}$$

$$\boxed{\frac{1}{g_1} + \frac{1}{g_2} = 2} \quad + \quad \boxed{g_1 = g_2 = 0}$$

this shows $g_1 = g_2 = 0$ is a solution because I took the reciprocal of both sides. That solution is excluded in

(#1) Locate the 8 cavities given on the handout on the stability diagram.

Work (3) as an example.

66a

We'll use $g_1 \equiv 1 - \frac{L}{R_1}$ & $g_2 \equiv 1 - \frac{L}{R_2}$

(1)  plane parallel

$R_1 = R_2 \to \infty$

$g_1 = g_2 = 1$ ✓

(5)  symmetric confocal

$R_1 = R_2 = L$

$g_1 = g_2 = 0$ ✓

More generally, a confocal cavity has:

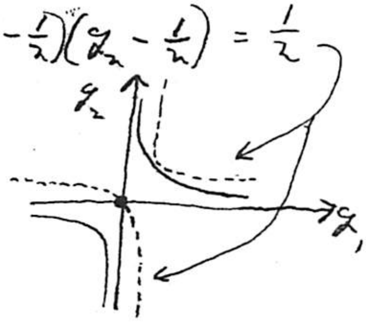
$R_1 + R_2 = 2L$

$\frac{1}{R_2} + \frac{1}{R_1} - \frac{2L}{R_1 R_2} = 0$

$\frac{L}{R_1} + \frac{L}{R_2} - \frac{2L^2}{R_1 R_2} = 0$

$g_1 + g_2 - 2g_1 g_2 = 0$

$2\left(g_1 - \frac{1}{2}\right)\left(g_2 - \frac{1}{2}\right) = \frac{1}{2}$



(6)  symmetric concentric

$R_1 = R_2 = \frac{L}{2}$

$g_1 = g_2 = -1$ ✓

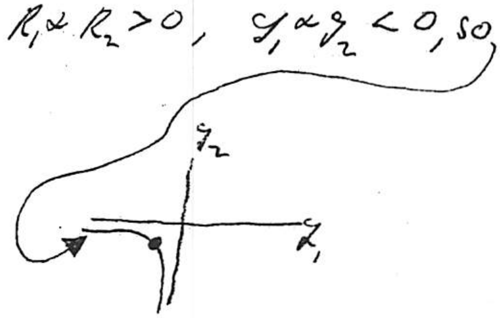
More generally, a concentric cavity has:

$R_1 + R_2 = L$

$\frac{1}{R_2} + \frac{1}{R_1} - \frac{L}{R_1 R_2} = 0$

$\frac{1 - g_1 g_2}{L} = 0 \Rightarrow g_1 g_2 = 1$ ✓

For R_1 & $R_2 > 0$, g_1 & $g_2 < 0$, so



(2)  "Between confocal and plane parallel"

$0 < \frac{L}{R_1} < 1$

$0 < \frac{L}{R_2} < 1$

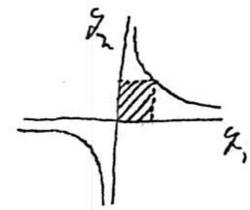
⇓

$-1 < -\frac{L}{R_1} < 0$

$-1 < -\frac{L}{R_2} < 0$

⇓

$\left.\begin{array}{l} 0 < g_1 < 1 \\ 0 < g_2 < 1 \end{array}\right\}$ ✓



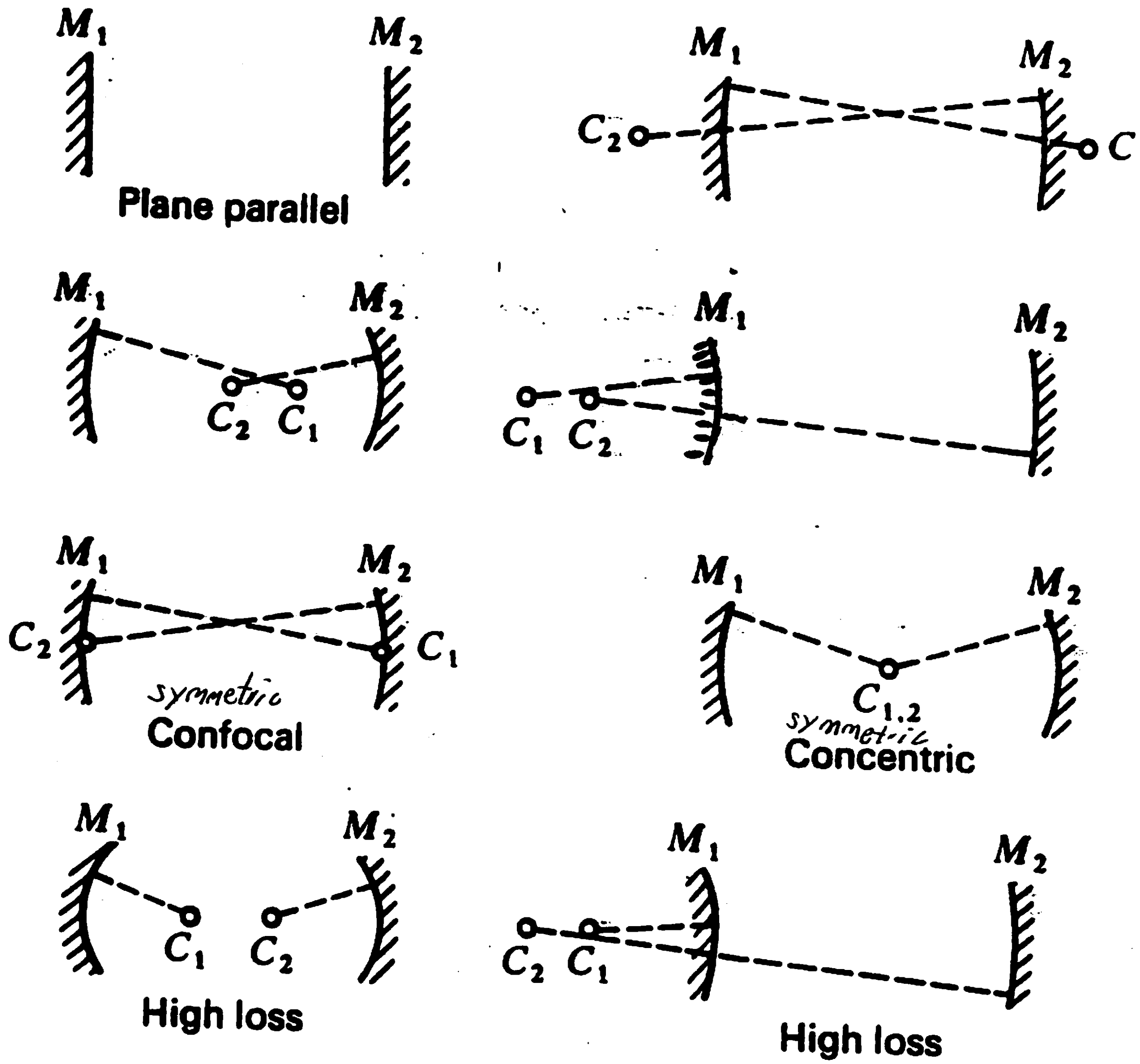
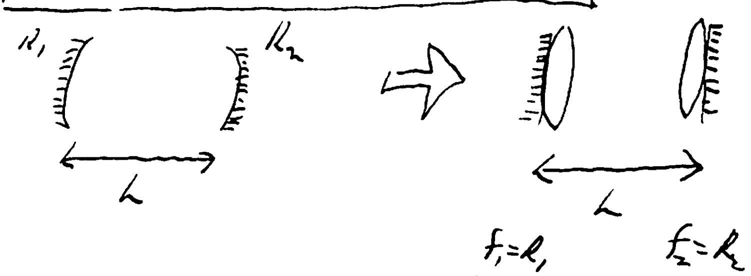
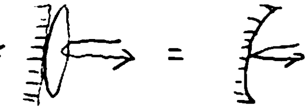


FIGURE 7.1 Examples of mirror configurations for optical masers. All except the bottom two exhibit low-loss resonant modes. Source: Reference 3.

Eigenmode of a linear cavity



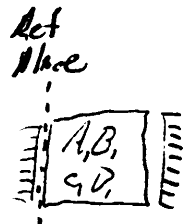
We need: 

2 lens hits = 1 mirror hit

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -\frac{2}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

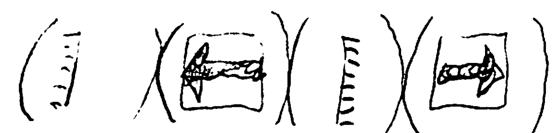
$$f = R \checkmark$$



$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ = 1 pass from left to right through $\square$

Includes f_1 and f_2 and anything else.

Note: $A_1 D_1 - B_1 C_1 = 1$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{RT} =$$ 

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_1 & B_1 \\ C_1 & A_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

$$= \begin{bmatrix} D_1 A_1 + B_1 C_1 & 2 B_1 D_1 \\ 2 A_1 C_1 & C_1 B_1 + A_1 D_1 \end{bmatrix}$$

$$D_1 A_1 + B_1 C_1 = D_1 A_1 + (A_1 D_1 - 1)$$

$$= \begin{bmatrix} 2 A_1 D_1 - 1 & 2 B_1 D_1 \\ 2 A_1 C_1 & 2 A_1 D_1 - 1 \end{bmatrix} \checkmark$$

<u>Note: $D = A$</u>

This ABCD matrix satisfies $q_1 = \frac{Aq_1 + B}{Cq_1 + D}$ where q_1 is the field at the reference plane in front of the (virtual) left flat end mirror.

> This next section uses the result from (notes p62) to find the field. The next section after that does it again using the approach in the text.
>
> Probably best to use the text version and skip this.

$$\frac{1}{q_1} = \underbrace{\frac{D-A}{2B}}_{\substack{=0 \text{ because} \\ D=A}} \pm i \frac{\sqrt{1-\left(\frac{D+A}{2}\right)^2}}{B}$$

(notes p62)

$$\underbrace{\left(\frac{D+A}{2}\right)^2}_{\text{because } D=A} = A^2 = AD = \underbrace{1+BC}_{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ is unitary}}$$

$$\frac{1}{q_1} = \pm i \frac{\sqrt{-BC}}{B} = \pm i\sqrt{-\frac{C}{B}} = \pm i\sqrt{-\frac{A_1C_1}{B_1D_1}}$$

sign of $\frac{A_1C_1}{B_1D_1}$ ⇨ $\frac{A_1C_1}{B_1D_1} = \frac{C_1^2}{D_1^2}\frac{A_1D_1}{B_1C_1} = \frac{C_1^2}{D_1^2}\frac{1+B_1C_1}{B_1C_1} = \frac{C_1^2}{D_1^2}\left(1+\frac{1}{B_1C_1}\right)$

Now $\frac{A+D}{2} = A = 2A_1D_1 - 1$

The stability criterion, then, is:

$$-1 \le 2A_1D_1 - 1 \le 1$$

$$0 \le A_1D_1 \le 1$$

$$-1 \le B_1C_1 \le 0$$

$$-\infty \le \frac{1}{B_1C_1} \le -1$$

so $\frac{A_1C_1}{B_1C_1} \le 0$

☆ Thus, q_1 is pure imaginary.

We must have:

$$\frac{1}{q_1} = -i\frac{\lambda}{\pi w_1^2 n}$$

so we choose the "−" solution for $1/q_1$:

$$\frac{1}{q_1} = -i\sqrt{-\frac{A_1C_1}{\quad}} \;///$$

Again following text

$$q_1 = \frac{A q_1 + B}{C q_1 + D} \Rightarrow C q_1^2 + (D - A) q_1 - B = 0$$

because $A = D$

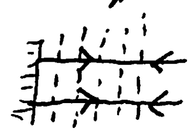
At plane mirror #1

choose sign that makes sense

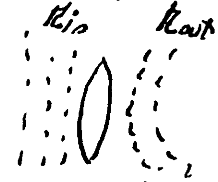
$$q_1 = \pm\sqrt{\frac{B}{C}} = \pm i\sqrt{-\frac{B}{C}} = \pm i\sqrt{-\frac{B_1 D_1}{A_1 C_1}} \checkmark$$

We saw (notes p 68) that $\frac{B_1 D_1}{A_1 C_1} < 0$ for a stable cavity,

So q_1 is pure imaginary $\Rightarrow$ flat phase fronts ✓



So we must have:



wave front impinging on imaginary plane mirror #1

wave fronts impinging on curved mirror #1

Now a lens transforms a spherical wave front according to

$$R_{out} = \frac{A R_{in} + B}{C R_{in} + D} \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$\frac{1}{R_{out}} = \frac{1}{R_{in}} - \frac{1}{f}$$

For $R_{out} = \infty$, $R_{in} = f = R_1$ ✓✓

For a stable linear cavity, the phase fronts must match the end mirror curvature.

70

Likewise for the right side, we end up with:

$$q_2 = -i\sqrt{-\frac{A_1B_1}{C_1D_1}} \quad \text{with} \quad \frac{A_1B_1}{C_1D_1} = \left(\frac{A_1}{D_1}\right)^2\left(\frac{B_1D_1}{A_1C_1}\right) < 0$$

so q_2 is also pure imaginary.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{RT} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} D_1 & B_1 \\ C_1 & A_1 \end{pmatrix}$$

$$= \begin{pmatrix} A_1D_1 + B_1C_1 & 2A_1B_1 \\ 2C_1D_1 & A_1D_1 + B_1C_1 \end{pmatrix}$$

$$= \begin{pmatrix} 2A_1D_1 - 1 & 2A_1B_1 \\ 2C_1D_1 & 2A_1D_1 - 1 \end{pmatrix}$$

$$q_2 = i\sqrt{-\frac{B}{C}}$$

$$q_2 = i\sqrt{-\frac{A_1B_1}{C_1D_1}}$$

Back to the 2-mirror resonator

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{pmatrix} f_2 \end{pmatrix}\begin{pmatrix} L \end{pmatrix}\begin{pmatrix} f_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1/R_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/R_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & \end{pmatrix}\begin{pmatrix} 1 - \frac{L}{R_1} & L \\ -\frac{1}{R_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L}{R_1} & L \\ -\frac{1}{R_2} + \frac{L}{R_1R_2} - \frac{1}{R_1} & 1 - \frac{L}{R_2} \end{pmatrix}$$

$$= \begin{pmatrix} g_1 & L \\ \frac{-(1-g_1g_2)}{L} & g_2 \end{pmatrix}$$

$$\frac{1}{q_1} = -i\sqrt{-\frac{A_1C_1}{B_1D_1}}$$

$$\mathrm{Im}\left(\frac{1}{q_1}\right) = -\cancel{\frac{\lambda}{\pi w_1^2}} = -\sqrt{-\frac{A_1C_1}{B_1D_1}}$$

$$w_1^2 = \frac{\lambda}{\pi}\sqrt{\frac{g_2L}{\left(\frac{1-g_1g_2}{L}\right)g_1}}$$

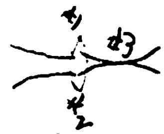
$$\boxed{w_1 = \left(\frac{L\lambda}{\pi}\right)^{1/2}\left(\frac{g_2}{g_1(1-g_1g_2)}\right)^{1/4} \quad \checkmark}$$

$$\boxed{w_2 = \left(\frac{L\lambda}{\pi}\right)^{1/2}\left(\frac{g_1}{g_2(1-g_1g_2)}\right)^{1/4} \quad \checkmark}$$

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi w^2 n}$$

$1/R = 0$ at the end mirrors

Now, the mode in the resonator is described by a single Gaussian mode so we just need to find where $z=0$ is and w_0.

Recall, we worked the example  and found:

$$\frac{w_3}{w_1} \quad \frac{\cancel{w_3}}{\cancel{w_{01}}} = \frac{f/z_{01}}{\sqrt{1+(f/z_{01})^2}} \qquad z_a = \frac{\pi w_{01}^2}{\lambda}$$

(notes p53) ⇨ Relabel:
$f \to f_1$ virtual lens #1
$w_3 \to w_0$ cavity waist
$w_1 \to w_1$ spot size mirror #1
$z_{0i} \to z_0$

$$g_1 = 1 - \frac{L}{R_1} = 1 - \frac{L}{f_1}$$

$$f_1 = \frac{L}{1-g_1}$$

$$\frac{w_0}{w_1} = \frac{f_1}{\sqrt{z_0^2 + f_1^2}} = \frac{f_1}{\left[\frac{\pi^2}{\lambda^2}\frac{L^2\lambda^2}{\pi^2}\left(\frac{g_2}{g_1(1-g_1g_2)}\right) + f_1^2\right]^{1/2}}$$

$$= \frac{L}{1-g_1}\,\frac{1}{L\left[\frac{g_2}{g_1(1-g_1g_2)} + \frac{1}{(1-g_1)^2}\right]^{1/2}} = \left[\frac{g_2(1-g_1)^2}{g_1(1-g_1g_2)} + 1\right]^{-1/2}$$

$$w_0 = \left[\frac{g_1(1-g_1g_2)}{g_2(1-g_1)^2 + g_1(1-g_1g_2)}\right]^{1/2}\left(\frac{L\lambda}{\pi}\right)^{1/2}\left(\frac{g_2}{g_1(1-g_1g_2)}\right)^{1/4}$$

$$= \left(\frac{L\lambda}{\pi}\right)^{1/2}\frac{\left[g_1g_2(1-g_1g_2)\right]^{1/4}}{\left(g_2 - 2g_1g_2 + \cancel{g_1^2g_2} + g_1 - \cancel{g_1^2g_2}\right)^{1/2}}$$

$$\boxed{w_0 = \left(\frac{L\lambda}{\pi}\right)^{1/2}\frac{\left[g_1g_2(1-g_1g_2)\right]^{1/4}}{\left[g_1+g_2-2g_1g_2\right]^{1/2}}} \quad ✓$$

ℓ = distance from lens to waist $= \dfrac{f}{1+\left(\frac{f}{z_{0}}\right)^2}$

= distance from left mirror to waist

$$= \frac{f_1}{1+\left(\frac{f_1}{z_0}\right)^2} \qquad z_0^2 = \frac{\pi^2}{\lambda^2} w_0^4 = \frac{\pi^2}{\lambda^2}\frac{L^2\lambda^2}{\pi^2}\frac{g_2}{g_1(1-g_1g_2)}$$

$$= \frac{f_1 z_0^2}{z_0^2+f_1^2} = \frac{\frac{L}{1-g_1} L^2 \frac{g_2}{g_1(1-g_1g_2)}}{\frac{L^2 g_2}{g_1(1-g_1g_2)} + \frac{L^2}{(1-g_1)^2}} = L \frac{\frac{g_2}{g_1(1-g_1)(1-g_1g_2)}}{\frac{g_2^2(1-g_1)^2 + g_1(1-g_1g_2)}{g_1(1-g_1g_2)(1-g_1)^2}}$$

$$= L \frac{g_2(1-g_1)}{g_2(1-g_1)^2 + g_1(1-g_1g_2)}$$

$$\boxed{\ell = L \frac{g_2(1-g_1)}{g_1+g_2-2g_1g_2}} \checkmark$$

Symmetric resonator

$R_1 = R_2 \equiv R$

$g_1 = g_2 = 1 - \frac{L}{R} \equiv g$

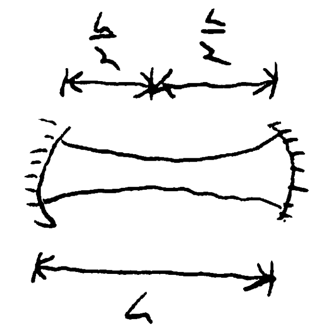
$$\omega_1 = \omega_2 = \omega = \sqrt{\frac{L\lambda}{\pi}} \left(\frac{1}{1-g^2}\right)^{1/4} \quad \checkmark$$

$$\omega_0 = \sqrt{\frac{L\lambda}{\pi}} \left[\frac{g^2(1-g^2)}{(2g-2g^2)^2}\right]^{1/4}$$

$(1+g)(1-g)$

$4g^2(1-g)^2$

$$= \sqrt{\frac{L\lambda}{\pi}} \left(\frac{1+g}{4(1-g)}\right)^{1/4} \quad \checkmark$$



$$\ell = L \, \frac{g(1-g)}{2g-2g^2} = \frac{L}{2} \quad \checkmark$$

$$\frac{\omega}{\omega_0} = \left(\frac{\frac{1}{1-g^2}}{\frac{1+g}{4(1-g)}}\right)^{1/4} = \left(\frac{4(1-g)}{(1+g)^2(1-g)}\right)^{1/4} = \left(\frac{4}{(1+g)^2}\right)^{1/4} \quad \checkmark$$

symmetric confocal $\Rightarrow$ $g = 0$ $\quad \omega_0 = \sqrt{\frac{L\lambda}{2\pi}} \quad \checkmark \quad \left(\approx \sqrt{\frac{(1m)(\frac{1}{2}\mu)}{2\pi}} \approx 300\mu\right)$

$$\frac{\omega}{\omega_0} = \sqrt{2} \quad \checkmark$$

Near plane parallel $\Rightarrow$ $R \gg L$ $\Rightarrow$ $g = 1-\varepsilon$ $\Rightarrow$ $\omega_0 = \sqrt{\frac{L\lambda}{\pi}} \left(\frac{1}{1-(1-\varepsilon)^2}\right)^{1/4} \approx \sqrt{\frac{L\lambda}{\pi}} \left(\frac{1}{2\varepsilon}\right)^{1/4} \quad \checkmark$

$$\frac{\omega_{near\ plane}}{\omega_{confocal}} = \frac{\sqrt{\frac{L\lambda}{\pi}} \left(\frac{1}{1-(1-\varepsilon)^2}\right)^{1/4}}{\sqrt{\frac{L\lambda}{2\pi}}\,\sqrt{2}} \approx \left(\frac{1}{1-(1-2\varepsilon)}\right)^{1/4}$$

$$\frac{\omega_{near\ plane}}{\omega_{confocal}} \approx \left(\frac{1}{2\varepsilon}\right)^{1/4}$$

and $\left(\frac{\omega}{\omega_0}\right)_{near\ plane} \approx \left(\frac{4}{4}\right)^{1/4} = 1$

$$\frac{\omega}{\omega_0} = \left[\frac{1}{1-g^2} \; \frac{4(1-g)}{1+g}\right]^{1/4} = \left[\frac{4}{(1+g)^2}\right]^{1/4} \approx \left(\frac{4}{4}\right)^{1/4} = 1 \quad \checkmark$$