Let's ignore the vector nature of $\vec{E}$ for this. We write:

Physical $E = Re(\tilde{E})$

$I \propto |\tilde{E}|^2$

$$\tilde{E}(x,y,z,t) = \tilde{E}(x,y,z)\, e^{i\omega t} \longrightarrow \text{time variation}$$

↑ complex amplitude varies in space

$$= E_0\, u(x,y,z)\, e^{-ikz}\, e^{i\omega t}$$

e^{-ikz} → propagates along z

so a plane wave would have $u = 1$.

text leaves this out, see eqs 4.6.4 and eqs 4.7.15a

$\boxed{\text{The lowest or Gaussian beam is given by}}$

$\boxed{\text{Paraxial approx again: close to a plane wave}}$

$$u(x,y,z) = u(r,z) = \frac{\omega_0}{\omega(z)}\, e^{-r^2/\omega^2(z)}\, e^{-ik\frac{r^2}{2R(z)}}\, e^{i\Phi(z)}$$

$r = \sqrt{x^2+y^2}$ = distance from z axis   Cylindrically symmetric (The higher order solutions are not.)

$$\omega(z) = \omega_0\sqrt{1+\frac{z^2}{z_R^2}} \quad \text{"spot size"}$$

$$z_R = \frac{\pi\omega_0^2}{\lambda} \quad \text{"Raleigh Range"} \quad (\text{I will inevitably write } z_0)$$

$$R(z) = z\left(1+\frac{z_R^2}{z^2}\right)$$

$$\Phi(z) = \tan^{-1}\frac{z}{z_R}$$

Everyone of these terms has an important and straightforward interpretation.

We'll spend some time discussing this. $u(r,z) = \text{mag}\; e^{i\,\text{phase}}$

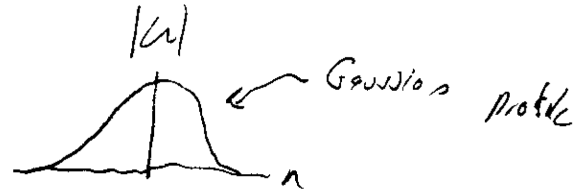
First note that $e^{-ik\frac{r^2}{2R(z)}}\, e^{i\Phi(z)}$ just affect the field phase

$$I \propto |\tilde{E}|^2 \propto |u|^2 = \frac{\omega_0^2}{\omega^2(z)}\, e^{-2r^2/\omega^2(z)}$$

So we just need to look at these terms to figure out the intensity profile.

$z = 0$

$$|u(x,y,z)| = e^{-r^2/\omega_0^2}$$



$\omega_0 = \frac{1}{e}$ field width

$z \neq 0$

$$|u(x,y,z)| = \frac{\omega_0}{\omega} e^{-r^2/\omega^2} \qquad \omega(z) > \omega_0$$

lower peak intensity

still Gaussian but with larger spot size

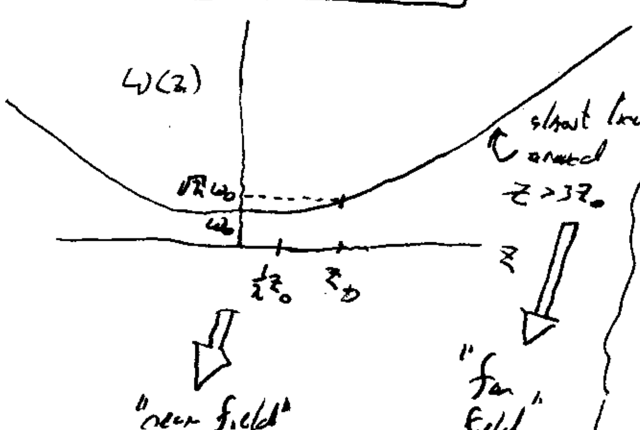
$D \equiv$ Intensity FWHM spot size defined by $\exp\left(\frac{-2r_{1/2}^2}{\omega^2}\right) = \frac{1}{2}$

$$\boxed{D(z) = \omega(z)\sqrt{2\ln 2}}$$

$$r_{1/2}^2 = -\tfrac{1}{2}\omega^2 \ln\tfrac{1}{2}$$

$$r_{1/2} = \omega\sqrt{\frac{\ln 2}{2}}$$

$$D = 2 r_{1/2}$$



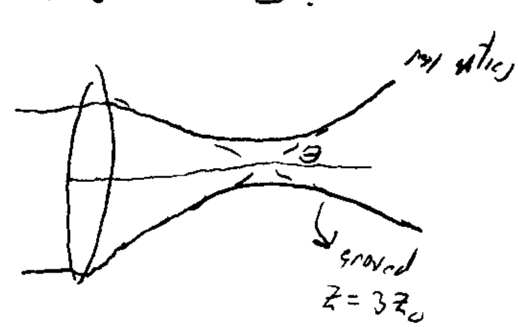
z_0 is distance for the spot size to reach $\sqrt{2}\,\omega_0$

or for $u(r=0)$ to reach $\frac{1}{\sqrt{2}}$

or " $I(r=0)$ to reach $\frac{1}{2} I_0$

$-z_0 \leq z \leq z_0$ is region over which the beam is "collimated."

$$z \to \infty \Rightarrow \omega(z) = \frac{\omega_0}{z_R} z = \frac{\lambda}{\pi \omega_0} z \ \checkmark$$



$$\theta_d \equiv \left.\frac{\omega}{z}\right|_{z\ \text{large}} = \frac{\lambda}{\pi \omega_0} \ \checkmark$$

d: divergence

Finally:

A Gaussian beam [illegible]

Numbers

$\lambda = \frac{1}{2}\mu m$

$w_0 = 1mm = 1000\mu$ $(D = 1.2mm)$

$$z_0 = \frac{\pi w_0^2}{\lambda} \approx 6m$$

$$\Theta_d = \frac{\lambda}{\pi w_0} = \frac{1}{6000} \approx 0.01^0$$

A real beam won't do as well

18,

Now let's look at the phase:

Remember: $E(\,)$ is close to a plane wave

$e^{i(\,)}$ are the corrections.

It helps to look at a spherical wave in the paraxial approx

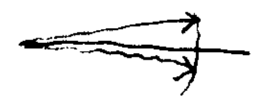
Recall: $E_{sp} \propto \dfrac{e^{-ikR}}{R}$ $\qquad R = \sqrt{x^2+y^2+z^2}$

$$R = z\sqrt{1+\frac{x^2+y^2}{z^2}}$$

In the paraxial approx: $r \ll z$

R = distance from point source
= radius of curvature of spherical phase front

$$\approx z + \frac{x^2+y^2}{2z}$$



$$R \approx z + \frac{r^2}{2R}$$

plane wave term

$$E_{sp} \propto \frac{e^{-ikz}\, e^{-i\frac{kr^2}{2R}}}{R}$$

correction

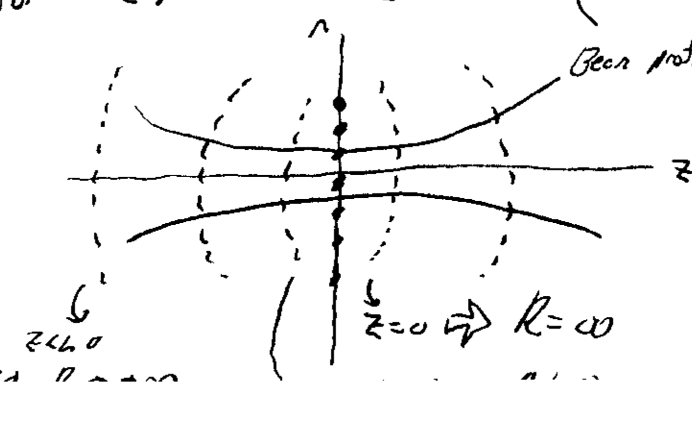
For our Gaussian:

$$U(\,) \propto e^{-ik\frac{r^2}{2R(z)}}$$

$$\boxed{\text{Recall: } E_{Gaussian} \propto e^{-ikz}\, U(\,)}$$

The surfaces of constant phase are ~spherical!
w/ radius of curvature R but R varies nontrivially w z.

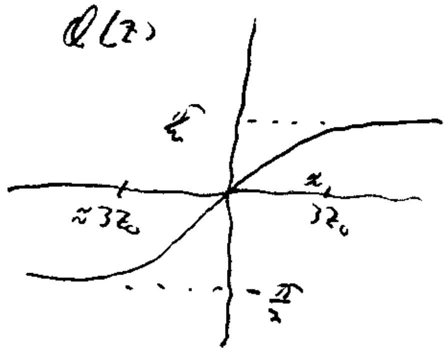
But $R \neq z$. $\qquad R(z) = z\left(1+\dfrac{z_R^2}{z^2}\right)$



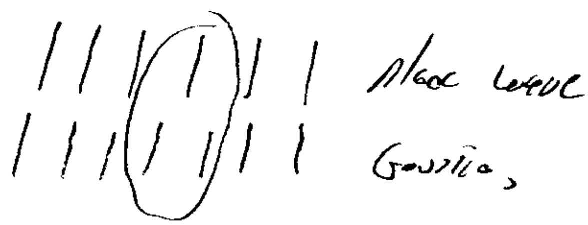
$\boxed{\text{More accurately: the lines of constant phase are parabolas}}$

$\boxed{\text{The beam profile is hyperbolic}}$

Finally: $e^{i\phi(z)} = e^{i\tan^{-1}\frac{z}{z_0}}$



As you go through a focus the phase fronts advance by π compared to a plane wave.



A Gaussian does not have a well defined k.
Only a plane wave has that.

Now, if you're in the far field, Gaussians are reasonably well understood using ray optics. In the near field where lasers live, ray optics gives the wrong answers.

Define $\frac{1}{q} = \underbrace{\frac{1}{R(z)}}_{\text{phase information}} - i \underbrace{\frac{\lambda}{\pi W^2(z)}}_{\text{intensity information}}$

$q(z)$ characterizes the beam at z.

$$\boxed{q_{out} = \frac{A q_{in} + B}{C q_{in} + D}}$$

ABCD rule for Gaussian beam propagation.

Similar to our rule for spherical phase fronts.

We can rewrite our Gaussian then as:
$u(r,\theta,z) = \frac{W_0}{W(z)} e^{-i\frac{r^2}{2q}} e^{i\phi}$

Propagation in air

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{air} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

$$q_{out} = q_{in} + L$$

(Recall for spherical waves we found $R_{out} = R_{in} + L$)

Lens

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{lens} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

$$q_{out} = \frac{q_{in}}{-\frac{q_{in}}{f} + 1}$$

$$\frac{1}{q_{out}} = \frac{1}{q_{in}} - \frac{1}{f} \quad \checkmark$$

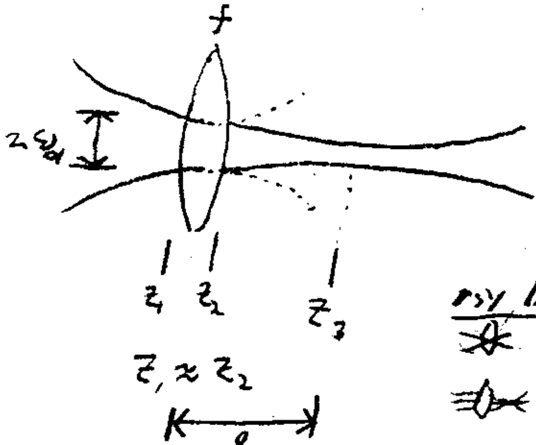
$w(z)$ is unchanged

$$\frac{1}{R_{out}} = \frac{1}{R_{in}} - \frac{1}{f}$$

Example (from Yariv/880 lecture notes p24)(Ex 4.6 in our text) S2

Input beam has a waist (ω_{01}) on a lens f.

Find location and size of output waist.



ray limit?

$z_2 = z_1$

$z_3 - z_2 = f$

⓪ Input beam ($z = z_1$)

$$\frac{1}{q_1} = \frac{1}{R(z_1)} - i\frac{\lambda}{\pi n \omega^2(z_1)}$$

$$= \frac{1}{\infty} - i\frac{\lambda}{\pi n \omega_{01}^2} = -i\frac{\lambda}{\pi n \omega_{01}^2}$$

Write ω_{01} in following as ω_1.

- $\omega_1 \rightarrow$ waist of input
- $\omega_2 \rightarrow$ spot size of output, $\omega_2 = \omega_1$ but not waist
- $\omega_3 \rightarrow$ waist of output

Lens ($z = z_2$)

① $$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} = -\frac{1}{f} - i\frac{\lambda}{\pi n \omega_{01}^2}$$

③ $$q_2 = \frac{1}{-\frac{1}{f} - i\frac{\lambda}{\pi n \omega_{01}^2}} = \frac{-\frac{1}{f} + i\frac{\lambda}{\pi n \omega_{01}^2}}{1/f^2 + \frac{\lambda^2}{\pi^2 n^2 \omega_{01}^4}}$$

$$= \frac{-a + ib}{a^2 + b^2}$$

let $a \equiv 1/f$

$b \equiv \frac{\lambda}{\pi n \omega_{01}^2}$

Displacement ($z = z_3$)

② $$\boxed{q_3 = q_2 + \ell} = \frac{-a}{a^2+b^2} + \ell + \frac{ib}{a^2+b^2}$$

④ $$\frac{1}{q_3} = \frac{1}{R_3} - i\frac{\lambda}{\pi n \omega_3^2} = \frac{\left(\frac{-a}{a^2+b^2} + \ell\right) - i\frac{b}{a^2+b^2}}{\left(\frac{-a}{a^2+b^2} + \ell\right)^2 + \left(\frac{b}{a^2+b^2}\right)^2}$$

Keep on board for next page

S3

Now, $R_3 = \infty$ so $\mathrm{Re}\left(\frac{1}{\tilde{q}_3}\right) = 0$ or

$z_0 = \frac{\pi W_0^2}{\lambda}$ $\quad z_0 \to z_{01}$, $W_0 \to W_1$ $\quad$ Don't confuse with z_1, z_2, z_3

$$\ell = \frac{a}{a^2+b^2} = \frac{\frac{1}{f}}{\frac{1}{f^2} + \frac{\lambda^2}{\pi n W_{01}^2}} = \frac{f}{1+\left(\frac{f}{\pi W_{01}^2 n/\lambda}\right)^2} = \frac{f}{1+\left(\frac{f}{z_{01}}\right)^2} \quad \checkmark$$

If z_{01} is large (input beam is a good approx to a collimated beam), $\ell = f$! (If z_{01} is small, $\ell \to$ small.)

$\boxed{\text{Look at } W_3}$

$$\mathrm{Im}\left(\frac{1}{\tilde{q}_3}\right) = -\frac{\lambda}{\pi n W_3^2} = \frac{\frac{-b}{a^2+b^2}}{\left(\frac{b}{a^2+b^2}\right)^2} = -\frac{a^2+b^2}{b} = -\frac{\frac{1}{f^2}+\left(\frac{\lambda}{\pi n W_{01}^2}\right)^2}{\frac{\lambda}{\pi n W_{01}^2}}$$

Using $\frac{-a}{a^2+b^2} + \ell = 0$ to simplify the denominator

$$W_3^2 = \frac{\frac{\lambda^2}{\pi n^2 W_{01}^2}}{\frac{1}{f^2} + \left(\frac{\lambda}{\pi n W_{01}}\right)^2} = \frac{f^2 \frac{\lambda^2}{\pi n^2 W_{01}^2}}{1+\left(\frac{f\lambda}{\pi n W_{01}^2}\right)^2}$$

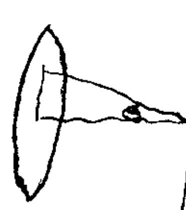
$$\frac{W_3^2}{W_{01}^2} = \frac{f^2 \frac{\lambda^2}{\pi^2 n^2 W_{01}^4}}{1+\left(\frac{f\lambda}{\pi n W_{01}^2}\right)^2}$$

$$\frac{W_3}{W_{01}} = \frac{f/z_{01}}{\sqrt{1+\left(f/z_{01}\right)^2}} \quad \checkmark$$

If $z_{01} \gg f$:

$$\frac{W_3}{W_{01}} \approx \frac{f}{z_{01}} = \frac{f\lambda}{\pi W_{01}^2}$$

$$W_3 \approx \frac{f\lambda}{\pi W_{01}} = \frac{1}{\pi}\left(\frac{f}{W_0}\right)\lambda = \frac{1}{\pi}\frac{\lambda}{\alpha}$$



F-number $= \frac{f}{\text{spot size on lens}} = \frac{f}{\text{lens diameter}}$

Higher order modes

$$u_{lm}(x,y,z) = \underbrace{\frac{w_0}{w} H_l\left(\frac{\sqrt{2}\,x}{w}\right) H_m\left(\frac{\sqrt{2}\,y}{w}\right)}_{\text{This is new: Hermite polynomials}} e^{-i\frac{kr^2}{2q}}\, e^{i(l+m+1)\phi}$$

contains both imaginary and real part

phase shift of $(l+m+1)\pi$ going through the focus instead of just π

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$I_{l,m=0}(x, y=0) =$

mode w/ l & m is called: TEM_{lm} (Transverse electro-magnetic)

See Yariv QE handout.

$$\frac{1}{q} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

$$e^{-i\frac{kr^2}{2R(z)} - \frac{k\lambda r^2}{2\pi w^2}}$$

$$u \propto e^{-r^2/w^2}$$

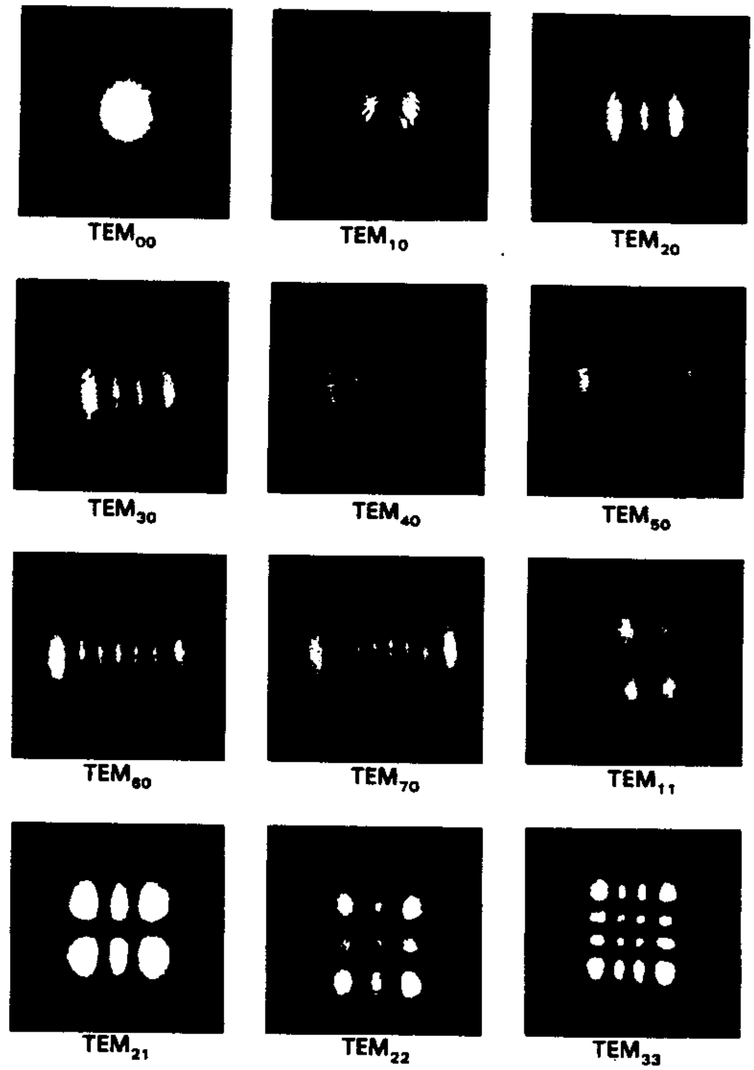


FIGURE 6.7 Some low-order optical-beam modes. *Source:* Reference 9.

and obtain directly from (2.2-14)

$$\psi_{l,m}(x, y) = E_{l,m}(\mathbf{r})e^{i\beta_{l,m}z} = E_0 H_l\left(\sqrt{2}\,\frac{x}{\omega}\right) H_m\left(\sqrt{2}\,\frac{y}{\omega}\right) \exp\left(-\frac{x^2 + y^2}{\omega^2}\right) \quad (6.10\text{-}4)$$

where H_l is the Hermite polynomial of order l. The eigenvalue $\beta_{l,m}$ is obtained from (2.2-8*a*) and (6.10-3)

$$\beta_{l,m} = k\left[1 - \frac{2}{k}\sqrt{\frac{n_2}{n}}\,(l + m + 1)\right]^{1/2} \quad (6.10\text{-}5)$$