Assume we have two semi-infinite media w/ a sharp flat interface that are: homogeneous, isotropic, linear, not time varying ($\varepsilon \neq \varepsilon(t)$)

{throws away a lot of interesting effects, but covers everyday optics & much technology}

Work w/ harmonic solutions in each medium (semi-infinite plane waves).

{get real fields by superposition}

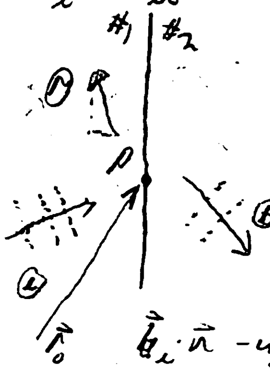
Assume incident, reflected, and transmitted waves.

{Can't generally match boundary conditions w/o reflected wave

→ If we're wrong will just get $\vec{E}_r = 0$. If we get things to work, uniqueness will exclude a 2nd transmitted beam.}

$$\vec{E}_i = \vec{E}_{i0}\, e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)} \qquad \vec{E}_r = \vec{E}_{r0}\, e^{i(\vec{k}_r\cdot\vec{r} - \omega_r t)} \qquad \vec{E}_t = \vec{E}_{t0}\, e^{i(\vec{k}_t\cdot\vec{r} - \omega_t t)}$$

↪ Just assuming they exist, space & time independent and linear.



$$\vec{E}_1 = \vec{E}_i + \vec{E}_r$$
$$\vec{E}_2 = \vec{E}_t$$

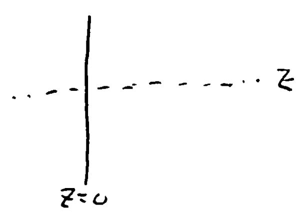
Say we match b.c. at $\vec{r}_0, t_0$.

At $t = t_0 + \Delta t$ (or $x = x_0 + \Delta x$, ...) the b.c. must still be satisfied, even all phases must change. For linear relationship between the fields, the phases must remain in synch:

Assign 9.15 from Griffiths for a proof

$$\vec{k}_i\cdot\vec{r} - \omega_i t = \vec{k}_r\cdot\vec{r} - \omega_r t = \vec{k}_t\cdot\vec{r} - \omega_t t$$

$\vec{r}$ & t are independent variables? {well, $\vec{r}$ must be on the interface. With no loss of generality, we can make $z=0$ the interface. $\vec{r} = x\hat{x} + y\hat{y}$ is now independent.



Suppose the ω's are different, and phases are equal at $(\vec{r}_0, t_0)$. Then they can't be equal for $(\vec{r}_0, t_0 + \delta t)$.

{Better: with no loss in generality, place origin at P. $\vec{r} = 0$ and...

so, $\boxed{\omega_i = \omega_r = \omega_t \equiv \omega}$ red light in gives you red light out.

$$v = \frac{\omega}{k} \qquad v_i k_i = v_r k_r = v_t k_t$$

$$\frac{c}{n_1}\frac{2\pi}{\lambda_1} = \frac{c}{n_1}\frac{2\pi}{\lambda_1} = \frac{c}{n_2}\frac{2\pi}{\lambda_2}$$

$$\boxed{n_1 \lambda_1 = n_2 \lambda_2}$$

$\lambda_1 \neq \lambda_2$, the larger index medium has the shorter wavelength (convention: "550 nm light" = $\lambda_{air} = 550$ nm)

$\vec{k}_i \cdot \vec{n} = \vec{k}_r \cdot \vec{n} = \vec{k}_t \cdot \vec{n} \Rightarrow k_{ix} x + k_{iy} y = k_{rx} x + k_{ry} y = \ldots$ 2-1

But, x & y are completely independent, so

Instead of setting $\vec{r}=0$ use our freedom to just set $y=0$

just set $x=0$

$k_{ix} x = k_{rx} x = k_{tx} x$

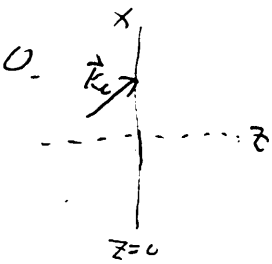
$k_{ix} = k_{rx} = k_{tx}$

$k_{iy} = k_{ry} = k_{ty}$

Since $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$

$|k_{iz}| = |k_{rz}| \neq |k_{tz}|$

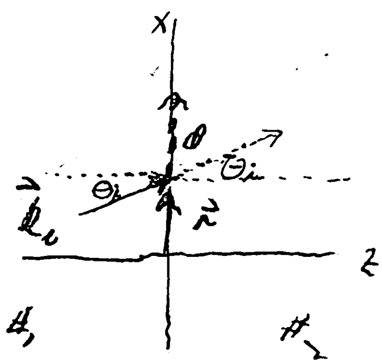
With no loss in generality, put $\vec{k}_i$ in xz plane: all k_y's $= 0$.



All $\vec{k}$'s are in xz plane.

$\boxed{\vec{k}_i, \vec{k}_r, \vec{k}_t \text{ are coplanar}}$

The common plane is the "plane of incidence"



z=0
interface

incident

$\vec{k}_i \cdot \vec{n} = k_i x \cos\theta = k_i x \cos(90° - \theta_i) = k_i x \sin\theta_i$

In optics, we almost always work with angles measured wrt the normal

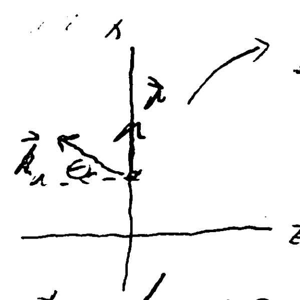
Note: usual convention is to draw tail of vector at point of consideration. People usually use the head so $\vec{k}_i$ is "in medium #1"

reflected

$\vec{k}_r \cdot \vec{n} = \vec{k}_i \cdot \vec{n}$

$k_i \sin\theta_i = k_r \sin\theta_r$

$\boxed{\theta_i = \theta_r \equiv \theta_1}$



This is the only way to draw this so that: $\vec{k}_r$ is coplanar, in medium #1, at angle θ_1

choose θ_r to be on opposite side of the normal in medium #1

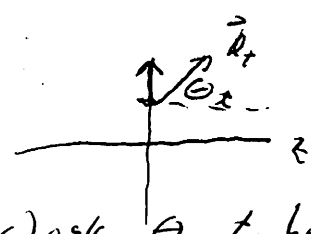
Transmitted

$k_i \sin\theta_i = k_t \sin\theta_t$

$\frac{1}{\lambda_1} \sin\theta_1 = \frac{1}{\lambda_2} \sin\theta_2$

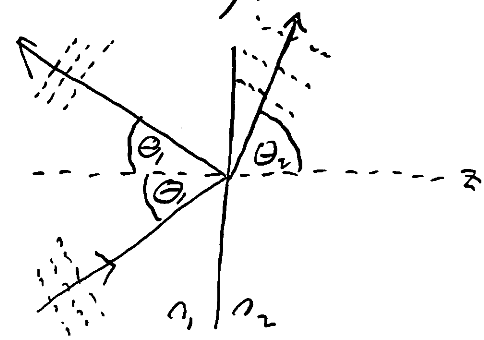
$\lambda_0 \frac{n}{n_1}$

$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$



choose θ_t to be on opposite side of the normal in medium #2

Here's a picture incorporating what we know for $n_1 > n_2$



$$\vec{E}_1 = \vec{E}_i + \vec{E}_r$$
$$\vec{E}_2 = \vec{E}_t$$

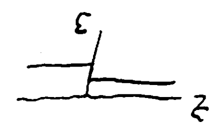
Phase continuity got us the geometrical behavior. To go further, we need MEs.

General $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ $\quad \vec{\nabla}\cdot\vec{B} = 0 \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

do conductors later — not Microscopic now

Specialized for media $\vec{\nabla}\cdot\vec{D} = \underbrace{\cancel{\rho_f}}_{=0} \quad \vec{\nabla}\cdot\vec{B} = 0 \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla}\times\vec{H} = \overset{=0}{\cancel{\vec{J}_f}} + \frac{\partial \vec{D}}{\partial t}$

$$\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P} \qquad \vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \vec{D} \equiv \varepsilon\vec{E} \qquad = \varepsilon\vec{E}$$



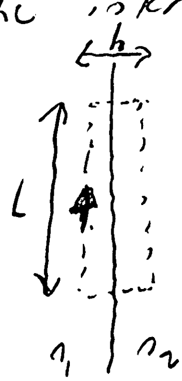
$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad \underbrace{\vec{M} = \chi_m \vec{H} \qquad \vec{H} \equiv \frac{1}{\mu}\vec{B}}_{\text{linear media}} \qquad \underbrace{= \frac{1}{\mu_0}\vec{B}}_{\text{optical media}}$$

can't pull out of derivative

$$\vec{\nabla}\cdot(\varepsilon\vec{E}) = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon\frac{\partial \vec{E}}{\partial t}$$

can pull this out of derivative

Last, to get b.c. we switch to integral form. We're not looking at local behavior but relationships across the interface, so integrals are better. Only need 2, but we'll do all 4.



$$\oint_C \vec{E}\cdot d\vec{\ell} = -\int_A \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Note: $A \neq A(t)$

- We only care about the interface, so let h be small: $\int_A \approx 0$

$$\vec{E}_1\cdot\vec{L}_1 + \vec{E}_2\cdot\vec{L}_2 = 0$$
$$\vec{E}_1\cdot\vec{L}_1 - \vec{E}_2\cdot\vec{L}_1 = 0$$
$$E_{1,tan} - E_{2,tan} = 0$$

tangent defined wrt the interface not e. up = positive

$$\boxed{E_{1,tan} = E_{2,tan}}$$

Figures based on any figure above

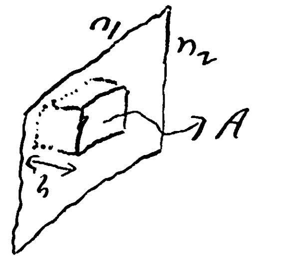
- Top & bottom don't contribute to $\oint_C$
- Let L be small (not zero) so $\vec{E}$ is constant. Large λ will do for any incident angle ($0°$ worst case).

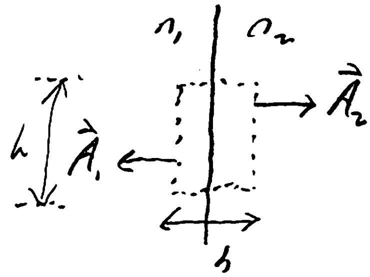
Possible Not Possible Not

$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_A \varepsilon \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$ ..... $\boxed{B_{1,tan} = B_{2,tan}}$

Div eqns will yield normal components.

$\oint_S \vec{D}\cdot d\vec{a} = 0$





Again, let h be small. Flux through sides is negligible, let $h \ll l$.

$\vec{D}_1\cdot\vec{A}_1 + \vec{D}_2\cdot\vec{A}_2 = 0$

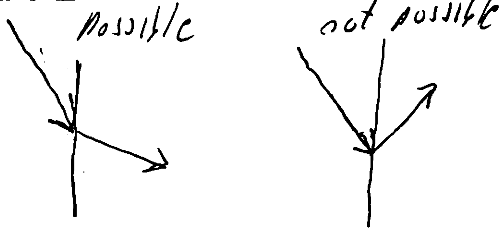
$\vec{D}_1\cdot\vec{A}_1 - \vec{D}_2\cdot\vec{A}_1 = 0$ because $\vec{A}_1 = -\vec{A}_2$

$D_{1n} - D_{2n} = 0$

$D_{1n} = D_{2n}$

$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$

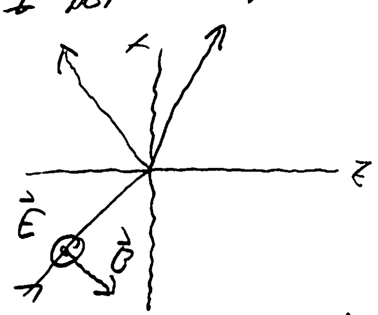
Likewise, $\oint_S \vec{B}\cdot d\vec{a} = 0 \Rightarrow \boxed{B_{1n} = B_{2n}}$



So, the normal and tangential b.c. for $\vec{E}$ are different (and would be for $\vec{B}$ if we used magnetic media - we won't).
Even more different when we treat conductors.

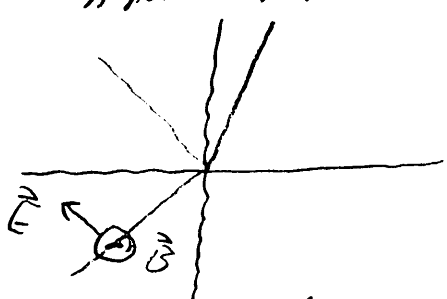
In general, $\vec{E}$ & $\vec{B}$ will have both components which have to be treated simultaneously. Instead, let's treat special cases where it's easier and use superposition to get the general case.

"⊥-pol" "s-pol"



① $\vec{E} \perp$ plane of incidence
② $\vec{E}$ is pure tangential
③ $\vec{B}$ is in plane & mixed

"∥-pol" "p-pol"



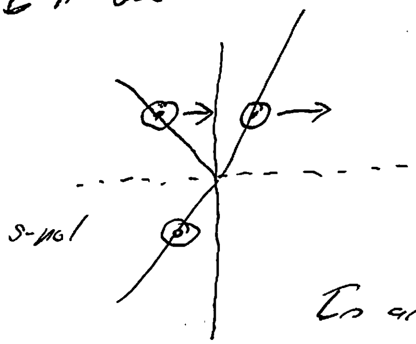
$\vec{B} \perp$ plane of incidence
$\vec{B}$ is pure tangential
$\vec{E}$ is in plane and mixed

Note: any two of the following will define the "plane of incidence":
$\vec{k}_i$ $\vec{k}_r$ $\vec{k}_t$ interface normal

usually easiest

If the incident wave is "s", then so are the output waves. Likewise for "p." This follows from MEs.

I'll just illustrate one case: "s" input, but the transmitted wave acquires a normal component.



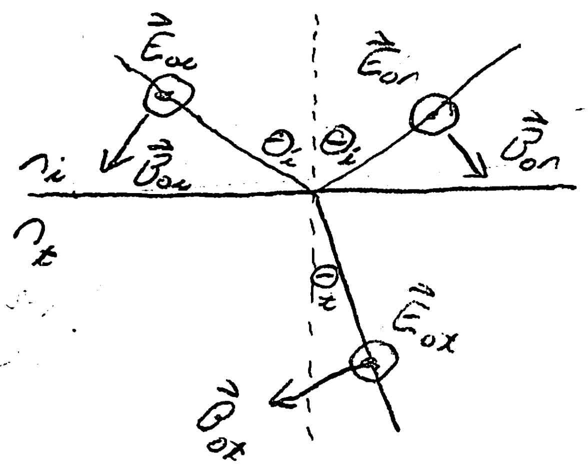
Note: $\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$ and $E_{1n} = \underbrace{\cancel{E_{in}}}_{=0} + E_{rn}$

so the reflected wave must also have a normal component.

In anycase, clearly $\vec{E}_t$ & $\vec{k}_t$ are not perpendicular.
Not possible!

Fresnel Equations (reflected/transmitted amplitudes as a function of $\vec{E}_{0i}$)

$\boxed{\text{s-pol} \\ \perp\text{-pol}}$

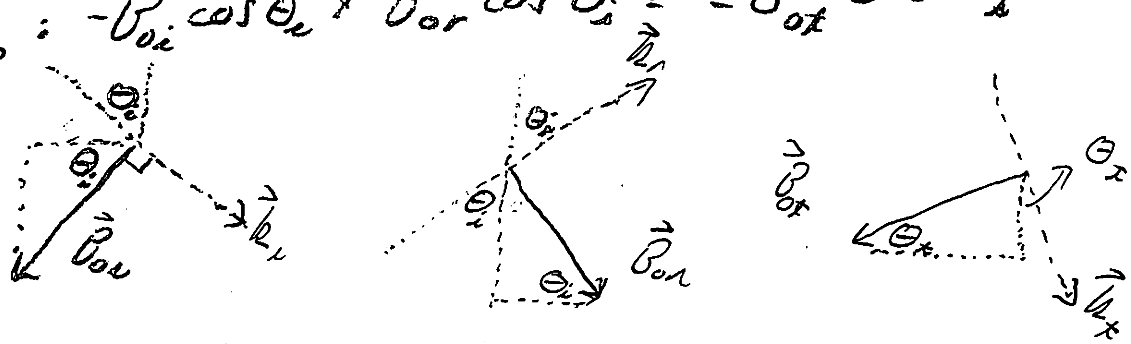


The field directions are given as they are at the interface.

I haven't shown they are all ⊙. We'll assume they are. If this is wrong, the derivation will tell us w/ negative signs.

E_{tan} : $E_{0i} + E_{0r} = E_{0t}$

B_{tan} : $-B_{0i}\cos\theta_i + B_{0r}\cos\theta_i = -B_{0t}\cos\theta_t$



$$B = \frac{E}{v} = \frac{n}{c}E$$

$$n_i\left(E_{0i} - E_{0r}\right)\cos\theta_i = n_t E_{0t}\cos\theta_t$$

It turns out ratios of E_{oi} are convenient so divide by E_{oi}

$$1 + \frac{E_{or}}{E_{oi}} = \frac{E_{ot}}{E_{oi}} \rightarrow \frac{E_{ot}}{E_{oi}} - \frac{E_{or}}{E_{oi}} = 1$$

$$\rightarrow n_t \cos\theta_t \underbrace{\frac{E_{ot}}{E_{oi}}}_{\equiv t_\perp} + n_i \cos\theta_i \underbrace{\frac{E_{or}}{E_{oi}}}_{\equiv r_\perp} = n_i \cos\theta_i$$

Two eqn in two unknowns: $t_\perp$, $r_\perp$

Solving (see Ex 3.4):

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

Given E_{oi}, θ_i, n_i & n_t we can use Snell's Law to get θ_t.

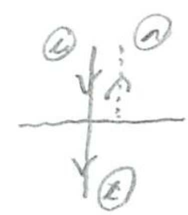
Thus, we have E_{or} & E_{ot} and, of course, the θ's.

Note: These are field ratios. We don't have the reflectivity, for example.

$r_\perp + t_\perp \neq 1$ or even $r_\perp^2 + t_\perp^2 \neq 1$

$r_\perp, t_\perp$ for air → glass is not the same as for glass → air.

Example 3.2 ⇨ using $n_g = 1.50$ and $\theta_i = 0$ ⇨ $\theta_t = 0$



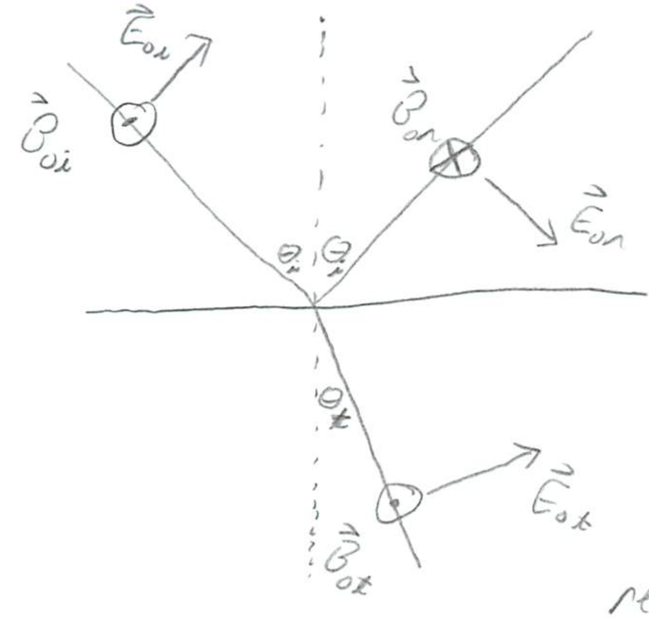
"normal incidence"

$$r_\perp = \frac{n_i - n_t}{n_i + n_t} \qquad t_\perp = \frac{2 n_i}{n_i + n_t}$$

air ($n_i = 1$) → glass ($n_t = 1.5$)		glass ($n_i = 1.5$) → air ($n_t = 1$)	
$r_\perp = -0.200$	$i \odot \otimes r$	$r_\perp = 0.200$	$i \odot \odot r$
$t_\perp = 0.800$	$t \odot$	$t_\perp = 1.20$	$t \odot$

like a string wave!!!

$\boxed{p\text{-pol},\ \parallel\text{-pol}}$



Why draw $\vec{B}_{0r}$ as ⊗?

We could have used ⊙. Either will do.

But, we focused on $\vec{E}$, not $\vec{B}$.

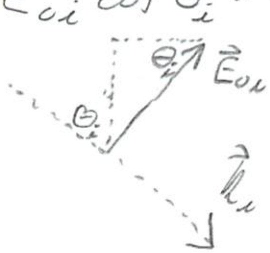
This choice keeps the $\vec{E}$'s pointing with positive tangential components in our reference figure.

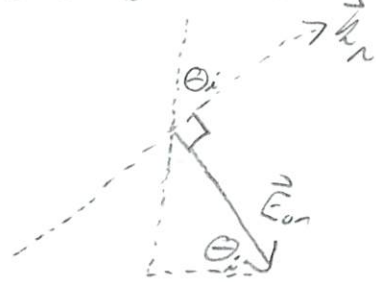
Since the $\perp$-case also had this, we'll get similar sign conventions: reflecting off a denser medium has $r < 0$.

B_{tan}: $\quad B_{0i} - B_{0r} = B_{0t}$

$$n_i E_{0i} - n_i E_{0r} = n_t E_{0t}$$

E_{tan}: $\quad E_{0i}\cos\theta_i + E_{0r}\cos\theta_i = E_{0t}\cos\theta_t$





$$n_t \frac{E_{0t}}{E_{0i}} + n_i \frac{E_{0r}}{E_{0i}} = n_i$$

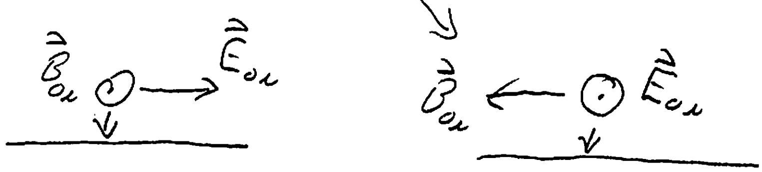
$$\cos\theta_t \underbrace{\frac{E_{0t}}{E_{0i}}}_{\equiv t_\parallel} - \cos\theta_i \underbrace{\frac{E_{0r}}{E_{0i}}}_{\equiv r_\parallel} = \cos\theta_i$$

$$r_{\parallel} = \frac{n_i \cos\theta_t - n_t \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}$$ Note "mixed indices"

$$t_{\parallel} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}$$

Repeating our example ⇒ normal incidence $\theta_i = 0 \rightarrow \theta_t = 0$

$$r_{\parallel} = \frac{n_i - n_t}{n_i + n_t} = r_{\perp} \qquad t_{\parallel} = \frac{2 n_i}{n_i + n_t} = t_{\perp}$$



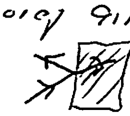
In either case, the electric field interaction w/ the interface is the same: $\vec{E}_i$ is pure tangential.

The result must be the same.

We'll spend a fair amount of time, now, trying to understand the behavior predicted by these equations.

Organizational questions: When are they min or max?
When do they change sign?
Difference $n_i > n_t$ and $n_i < n_t$

Terminology: $n_i < n_t$ "going from fast to slow"
"external incidence" text uses this a lot
think going air → glass.



Brewster's Angle

$r_\parallel$ can be zero at θ_B, internal or external incidence.

$$r_\parallel = \frac{n_i \cos\theta_t - n_t \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} = 0$$

$n_i \cos\theta_t = n_t \cos\theta_i$ but $n_i = \dfrac{n_t \sin\theta_t}{\sin\theta_i}$ (Snell's Law)

$\cancel{n_t} \sin\theta_t \cos\theta_t = \cancel{n_t} \cos\theta_i \sin\theta_i$ but $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\theta_t = \sin 2\theta_i \Rightarrow$

$2\theta_t = 2\theta_i$ No

$2\theta_t = \pi - 2\theta_i$ ✓

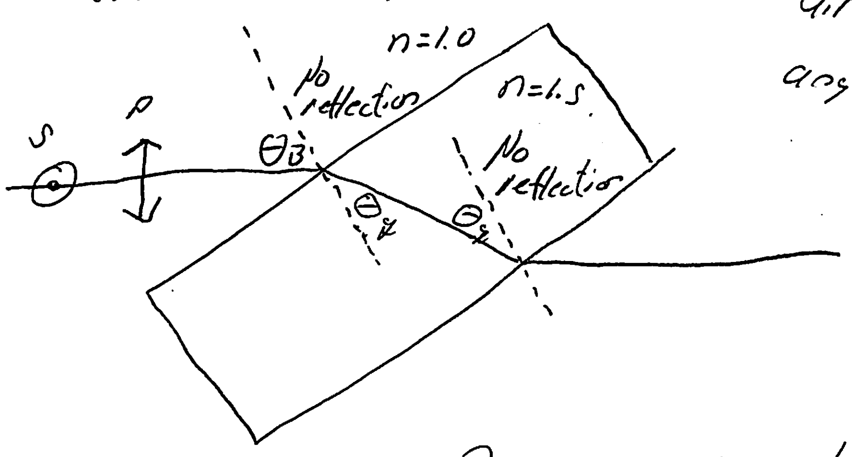
only soln keeping $0 \le \theta_t \le 90°$

$\theta_i + \theta_t = \frac{\pi}{2}$

$n_i \sin\theta_i = n_t \sin\left(\frac{\pi}{2} - \theta_i\right) = n_t \cos\underbrace{\theta_i}_{\equiv\theta_B}$

$$\boxed{\tan\theta_B = \frac{n_t}{n_i}}$$ p-pol only

A surface angled at Brewster's angle will not reflect p-pol - no reflection losses!



air to glass: $\tan\theta_B = \frac{1.5}{1.0}$ $\boxed{\theta_B = 56.3°}$

angle in glass:

$n_{air} \sin\theta_{air} = n_g \sin\theta_g$

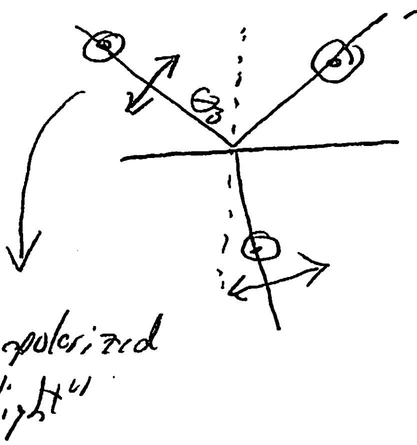
$\sin\theta_g = \dfrac{n_{air} \sin\theta_{air}}{n_g}$

$= \dfrac{\sin\theta_B}{1.5}$

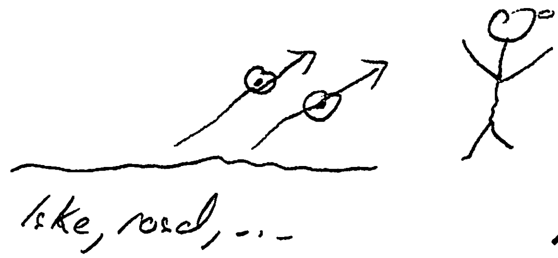
$\boxed{\theta_g = 33.7°}$

Brewster angle glass to air: $\tan\theta_B' = \frac{1.0}{1.5}$ $\boxed{\theta_B' = 33.7°}$

Polarization by reflection



As we'll see, if you're close to θ_B, you'll still get a significant effect.



"Horizontal polarization"

Some sunglasses use filters that only pass vertical polarization: "polarizers"

Note:
"s" or "p" relative to interface
"horizontal", "vertical" relative to the floor

Total Internal Reflection

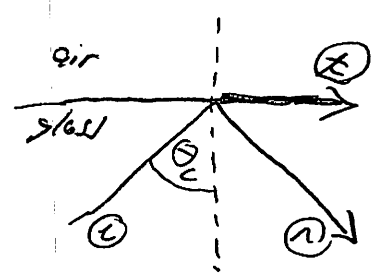
(complete reflection for $\theta > \theta_c$, internal incidence)
$\hookrightarrow$ well, this is subtle, as we'll see

$$n_t \sin\theta_t = n_i \sin\theta_i$$

$$\sin\theta_t = \underbrace{\frac{n_i}{n_t}}\sin\theta_i$$

this can exceed 1 for internal incidence where $n_i > n_t$ (glass $\to$ air).

Define: $\theta_i \equiv \theta_c$ when $\sin\theta_t = 1$ or $\theta_t = 90°$



$$\boxed{\theta_c = \frac{n_t}{n_i}}$$

I'll just cite results from Text:

For $\theta_i > \theta_c$, TIR

So this happens for grazing incidence.

s-pol, ⊥-pol $\Big\}$ $E_{or} = E_{oi}\, e^{i\varphi_\perp}$ $\qquad \varphi_\perp = 2\tan^{-1}\left(\frac{n_t}{n_i}\alpha\right)$

so $|E_{or}| = |E_{oi}|$ $\qquad \alpha \equiv \dfrac{\sqrt{\left(\frac{n_i}{n_t}\sin\theta_i\right)^2 - 1}}{\cos\theta_i}$

p-pol, ∥-pol $\Big\}$ $E_{or} = E_{oi}\, e^{i\varphi_\parallel}$ $\qquad \varphi_\parallel = 2\tan^{-1}\left(\frac{n_i}{n_t}\alpha\right)$

{same α}

There's no transmitted wave but there are fields in the n_t medium as we'll see later