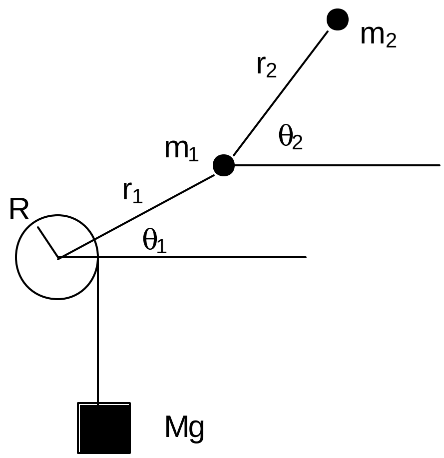
Derivation of Compound Trebuche Equations of Motion

(Requires a College Sophomore Mechanics Course)



The mass m_1 has been used to model the moment of inertia of the arm and turning shaft. Any potential energy of the arm has been ignored. The taut sling r_2 tethers the throwing mass m_2. The dropped weight is of course has mass M.

In Cartesian coordinate the position of m_1 and m_2 are given by the expressions:

$$x_1 = r_1 \sin(q_1) \quad \text{and} \quad x_1 = r_1 \sin(q_1) + r_2 \sin(q_2)$$
$$y_1 = r_1 \cos(q_1) \qquad\qquad y_1 = r_1 \cos(q_1) + r_2 \cos(q_2)$$

One can then calculate the Kinetic Energy (K) and the Potential Energy (U) of the system.

$$K = \frac{1}{2} m_1 r_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 (r_1^2 \dot{q}_1^2 + r_2^2 \dot{q}_2^2 + 2 r_1 r_2 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2)) + \frac{1}{2} M R^2 \dot{q}_1^2$$
$$U = -Mghq_1 + m_2 g (r_1 \sin(q_1) + r_2 \sin(q_2))$$

The Lagrangian is given by L=K-U, and the equations of motion can be found by variational techniques using the Euler-Lagrange equations.

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 0 \quad \text{and} \quad \frac{\partial L}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = 0$$

The resulting equations of motion respectively are:

$$Mgh - m_2 g r_1 \cos(q_1) - m_2 \dot{q}_2^2 r_1 r_2 \sin(q_1 - q_2) - m_2 r_1^2 \ddot{q}_1 - m_1 r_1^2 \ddot{q}_1 - M r^2 \ddot{q}_1 - m_2 r_1 r_2 \ddot{q}_2 \cos(q_1 - q_2) = 0$$

and

$$- m_2 g r_2 \cos(q_2) + m_2 \dot{q}_1^2 r_1 r_2 \sin(q_1 - q_2) - m_2 r_2^2 \ddot{q}_2 - m_2 r_1 r_2 \ddot{q}_1 \cos(q_1 - q_2) = 0$$

These coupled differential equations are very difficult if not impossible to solve in closed form. These are solved numerically with a computer by setting initial conditions.

$q_1 = -p/2, q_2 = -p/2, \dot{q}_1 = \dot{q}_2 = 0$ at t=0 seconds.

One is the left with two equations in two unknowns $\ddot{q}_1$ and $\ddot{q}_2$. Solving for $\ddot{q}_1$ and $\ddot{q}_2$ these are plugged back into the equations.

$$t' = t + \Delta t$$

$$q_1' = q_1 + \dot{q}_1 \Delta t$$

$$q_2' = q_2 + \dot{q}_2 \Delta t$$

$$\dot{q}_1' = \dot{q}_1 + \ddot{q}_1 \Delta t$$

$$\dot{q}_2' = \dot{q}_2 + \ddot{q}_2 \Delta t$$

and the process is repeated iteratively.