Derivation of Compound Trebuche Equations of Motion

(Requires a College Sophomore Mechanics Course)

The mass $m_1$ has been used to model the moment of inertia of the arm and turning shaft. Any potential energy of the arm has been ignored. The taut sling $r_2$ tethers the throwing mass $m_2$. The dropped weight is of course has mass $M$.

In Cartesian coordinate the position of $m_1$ and $m_2$ are given by the expressions:

$$x_1 = r_1 \sin(\theta_1) \quad \text{and} \quad x_1 = r_1 \sin(\theta_1) + r_2 \sin(\theta_2)$$

$$y_1 = r_1 \cos(\theta_1) \quad \text{and} \quad y_1 = r_1 \cos(\theta_1) + r_2 \cos(\theta_2)$$

One can then calculate the Kinetic Energy ($K$) and the Potential Energy ($U$) of the system.

$$K = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (r_1^2 \dot{\theta}_1^2 + r_2^2 \dot{\theta}_2^2 + 2r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) + \frac{1}{2} MR^2 \dot{\theta}_1^2$$

$$U = -Mgh \theta_1 + m_2 g (r_1 \sin(\theta_1) + r_2 \sin(\theta_2))$$

The Lagrangian is given by $L = K - U$, and the equations of motion can be found by variational techniques using the Euler-Lagrange equations.

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) = 0 \quad \text{and} \quad \frac{\partial L}{\partial \theta_2} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) = 0$$

The resulting equations of motion respectively are:

$$Mgh - m_2 gr_1 \cos(\theta_1) - m_2 r_2 \dot{\theta}_2 r_2 \sin(\theta_1 - \theta_2) - m_2 r_1 \dot{\theta}_1 - m_2 r_1 \dot{\theta}_1 - MR^2 \dot{\theta}_1 - m_2 r_1 r_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) = 0$$

and

$$-m_2 gr_2 \cos(\theta_2) + m_2 r_2 \ddot{\theta}_1 r_2 \sin(\theta_1 - \theta_2) - m_2 r_2 \dot{\theta}_2 - m_2 r_2 \dot{\theta}_2 - m_2 r_2 \dot{\theta}_2 - m_2 r_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) = 0$$
These coupled differential equations are very difficult if not impossible to solve in closed form. These are solved numerically with a computer by setting initial conditions.

\[ \theta_1 = -\frac{\pi}{2}, \theta_2 = -\frac{\pi}{2}, \dot{\theta}_1 = \dot{\theta}_2 = 0 \quad \text{at } t=0 \text{ seconds.} \]

One is the left with two equations in two unknowns \( \theta_1 \) and \( \theta_2 \). Solving for \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \) these are plugged back into the equations.

\[
\begin{align*}
  t' &= t + \Delta t \\
  \theta_1' &= \theta_1 + \dot{\theta}_1 \Delta t \\
  \theta_2' &= \theta_2 + \dot{\theta}_2 \Delta t \\
  \dot{\theta}_1' &= \dot{\theta}_1 + \ddot{\theta}_1 \Delta t \\
  \dot{\theta}_2' &= \dot{\theta}_2 + \ddot{\theta}_2 \Delta t
\end{align*}
\]

and the process is repeated iteratively.