Lecture 4
Propagation of Errors

1) Introduction

Suppose, for example, we measure the current (I) and the resistance (R) of a certain piece of electrical equipment (e.g. a resistor). The voltage (V) drop across the resistor can be calculated using Ohm's law:

$$V = IR$$

If we know the uncertainties (e.g. standard deviations) in the measured variables (I and R) what is the uncertainty in V?

More formally we could ask, given that we have a functional relationship between several measured variables (x, y, z), i.e.

$$Q = f(x, y, z)$$

What is the uncertainty in Q if the uncertainties in x, y, and z are known? Usually when we talk about uncertainties in a measured variable such as x we usually assume that the value of x represents the mean of a Gaussian distribution and the uncertainty in x is the standard deviation (σ) of the Guassian distribution. A word of caution here, not all measurements can be represented by Gaussian distributions, but more on that later!

To answer this question we use a technique called Propagation of Errors.

2) Propagation of Error Formula:

To calculate the variance in Q as a function of the variances in x and y we use the following:

$$\sigma_Q^2 = \sigma_x^2(\partial Q/\partial x)^2 + \sigma_y^2(\partial Q/\partial y)^2 + 2\sigma_{xy}(\partial Q/\partial x)(\partial Q/\partial y)$$

Note: if the variables x and y are uncorrelated (σ_{xy} =0) then the last term in the above equation is 0.

We can derive the above formula as follows:
Assume we have several measurement of the quantities x (e.g. x_1, x_2...x_i) and y (e.g. y_1, y_2...y_i).
We can calculate the average of x and y using:

$$\mu_x = \sum_{i=1}^{N} x_i / N \text{ and } \mu_y = \sum_{i=1}^{N} y_i / N$$

Let's define: $Q_i + f(x_i, y_i)$

$Q + f(\mu_x, \mu_y)$ evaluated at the *average* values

Now expand Q_i about the average values:

$$Q_i = f(\mu_x, \mu_y) + (x_i - \mu_x)\left.\frac{\partial Q}{\partial x}\right|_{\mu_x} + (y_i - \mu_y)\left.\frac{\partial Q}{\partial y}\right|_{\mu_y} + \text{higher order terms}$$

Assume that we neglect the higher order terms (i.e. the measured values are close to the average values). We can rewrite the above as:

$$Q_i - Q = (x_i - \mu_x)\left.\frac{\partial Q}{\partial x}\right|_{\mu_x} + (y_i - \mu_y)\left.\frac{\partial Q}{\partial y}\right|_{\mu_y}$$

We would like to find the variance of Q. By definition the variance of Q is just:

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}(Q_i - Q)^2$$

If we expand the summation using the definition of Q - Q_i we get:

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_x)^2\left(\frac{\partial Q}{\partial x}\right)^2_{\mu_x} + \frac{1}{N}\sum_{i=1}^{N}(y_i-\mu_y)^2\left(\frac{\partial Q}{\partial y}\right)^2_{\mu_y} + \frac{2}{N}\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}$$

Since the derivatives are all evaluated at the average values (μ_x, μ_y) we can pull the derivatives outside of the summations. Finally, remembering the definition of the variance we can write:

$$\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)^2_{\mu_x} + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)^2_{\mu_y} + \frac{2}{N}\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)$$

If the measurements are uncorrelated then the summation in the above equation will be very close to zero (if the variables are truly uncorrelated then the sum is 0) and can be neglected. Thus for uncorrelated variables we have:

$$\boxed{\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)^2_{\mu_x} + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)^2_{\mu_y}}$$ <u>uncorrelated errors</u>

If however x and y are correlated, then we define σ_{xy} using:

$$\sigma_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)$$

The variance in Q including correlations is given by:

$$\boxed{\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)^2_{\mu_x} + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)^2_{\mu_y} + 2\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}\sigma_{xy}}$$ <u>correlated errors</u>

<u>EXAMPLE:</u> Power in an electric circuit. The power in an electric circuit can be calculated from:

$$P = I^2R$$

Let $I = 1.0 \pm 0.1$ amp and $R = 10 \pm 1\ \Omega$. The power is then 10 watts. The variance in the power can be calculated using propagation of errors (uncorrelated case here since only σ_x and σ_y are given).

$$\sigma_P^2 = \sigma_I^2\left(\frac{\partial P}{\partial I}\right)^2_{I=1} + \sigma_R^2\left(\frac{\partial P}{\partial R}\right)^2_{R=10} = \sigma_I^2(2IR)^2 + \sigma_R^2\left(I^2\right)^2 = (0.1)^2(2\cdot 1\cdot 10)^2 + (1)^2(1^2)^2 = 5\ \text{watts}^2$$

Sometimes its convenient to put the above calculation in terms of relative errors:

$$\frac{\sigma_P^2}{P^2} = \frac{\sigma_I^2}{P^2}\left(\frac{\partial P}{\partial I}\right)^2 + \frac{\sigma_R^2}{P^2}\left(\frac{\partial P}{\partial R}\right)^2 = \frac{4\sigma_I^2}{I^2} + \frac{\sigma_R^2}{R^2} = 4\left(\frac{0.1}{1}\right)^2 + \left(\frac{1}{10}\right)^2$$

The beauty of this form is that it immediately tells us that the uncertainty in the *current* dominates the uncertainty in the power. This shows that in order to make a substantial decrease in the variance some variables (current here!) must be measured more accurately than others.

3) <u>Problems with this method:</u>
In calculating the variance using propagation of errors we usually assume that the we are dealing with <u>Gaussian</u> like errors for the measured variable (e.g. x). Unfortunately, just because x is described by a Gaussian distribution does not mean that $f(x)$ will be described by a Gaussian distribution!

I) *What does the standard deviation that we calculate from propagation of errors mean?*

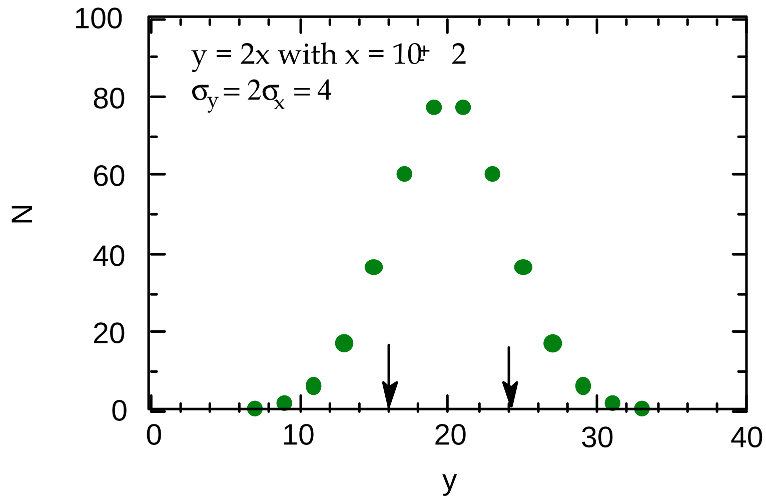
a) Example when the new distribution is Gaussian.
Let $f(x) = Ax$, with A = a constant and x a Gaussian variable. The probability distribution function for x is by definition:

$$p(x,\mu_x,\sigma_x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

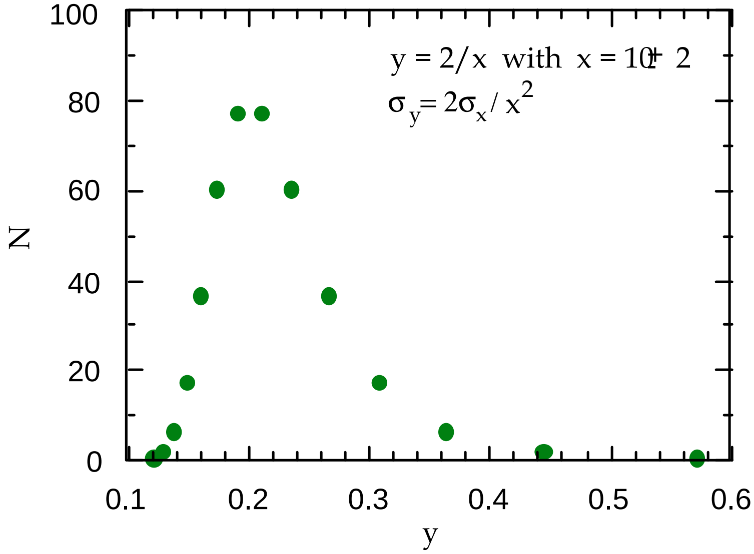
it also follows that $\mu_f = A\mu_x$ and $\sigma_f = A\sigma_x$. Putting this into the above equation we have:

$$p(x,\mu_x,\sigma_x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} = \frac{1}{\frac{\sigma_f}{A}\sqrt{2\pi}} e^{-\frac{(\frac{f}{A}-\frac{\mu_f}{A})^2}{2\left(\frac{\sigma_f}{A}\right)^2}} = \frac{A}{\sigma_f\sqrt{2\pi}} e^{-\frac{(f-\mu_f)^2}{2\sigma_f^2}} = A\cdot p(f,\mu_f,\sigma_f)$$

Thus the new function which is the probability distribution function for f, $p(f, \mu_f, \sigma_f)$ is also described by a Gaussian probability distribution function.



b) Example when the new distribution is non-Gaussian:
Just let $f(x) = A/x$.
The transformed probability distribution function for f does not have the form of a Gaussian.



$P(y > 5\sigma_y) = 0.5\%$ c.f. $3\times10^{-5}\%$ for Gaussian

The moral of the story here is that standard deviations that are calculated using propagation of errors are not always the standard deviation that we associate with a Gaussian distribution (e.g. 68% of area within ±1 standard deviation).

II) *Unphysical situations can arise if we use the propagation of errors results blindly!*

Example:

Suppose we measure the volume of a cylinder: $V = \pi R^2 L$.
Let R = 1 cm exact, and L = 1.0 ± 0.5 cm.
Using propagation of errors we have: $\sigma_V = \pi R^2 \sigma_L = \pi/2$ cm^3.

and <u>$V = \pi \pm \pi/2$ cm^3</u>

However, if the error on V (σ_V) is to be interpreted in the Gaussian sense then the above result says that there's a finite probability (• 3%) that the volume (V) is < 0 since V is only two standard deviations greater than 0! Clearly this is unphysical and care must be taken in interpreting the meaning of σ_V.